Technische Universität München

Lehrstuhl für Theoretische Informationstechnik

Frames for the Space of Multi-banded Finite-Energy Signals on Locally Compact Abelian Groups with Applications to Sampling Problems

Master's Thesis

Ezra Tampubolon
Mtr.-Nr. 3601599

Advisor Dr.-Ing. Volker Pohl
Examiner Univ.-Prof. Dr.-Ing. Dr.rer.nat. Holger Boche

Contents

Abstract

This thesis concerns with redundant and non-redundant multi-channel non-uniform periodic sampling and reconstruction of square-integrable multi-banded signals on locally compact Abelian groups, involving linear preprocessors (in which some practical relevant domains are contained, e.g. cartesian product of $\mathbb{R}$, of $\mathbb{Z}$, of $\mathbb{Z}/k\mathbb{Z}$). In particular, this problem is connected to the problem of finding a frame of exponentials weighted by functions for the frequency space of the signal of interest. We give necessary and sufficient conditions on the corresponding collection of functions s.t. it forms this class of frames. Some particular examples are also given.

Furthermore, this thesis provides an extension of the method of recovering continuous 1-dim. signals from modulus of its Fourier measurements in infinite dimensional spaces given in [60, 48] to continuous 2-dim. signals in infinite dimensional spaces. We shall show that 8 times the 2-dim. Nyquist rate, instead of expected 16 times is sufficient to reconstruct compactly supported signals, up to some exceptions, from its fourier measurements.

Acknowledgements

It has been a pleasure working with the chair for theoretical information technology (LTI) in Munich for about three years. I would like to express my profound and sincere gratitude to my advisors Dr. Volker Pohl and Prof. Holger Boche for their constant supports, the useful comments, remarks, engagement, and patience through the learning process of this master thesis. Foremost, i would like thank Dr. Volker Pohl for providing his time and energy to supervise me all the years, through the Bachelor's thesis, seminar work, and research project. I owe far more than i could put into words.

Furthermore, i would like to thank Gisbert Janßen for introducing me the interesting field of quantum communication, and supervising me in my seminar work in the corresponding course. Also i would thank to everyone else in LTI, especially Dr. Philipp Walk, Dr. Janis Nötzel, and for my "room mates" (in chronological order) Fanny Yang, Cagkan Yapar, Andrea Grigorescu, Anna Frank, and Stephan Kaltenstadler.

Furthermore, i would also like to thank all my friends, especially: Christoph Klose, Jakob Jungmaier, Phillip Kosterhon, Matthias Pegert, Johannes Keller.

Finally, I take this opportunity to express the profound gratitude from my deep heart to my beloved parents, Ebenezer Tampubolon and Ratna Tambunan, my brother, Amos Tampubolon, and my girlfriend, Regine Hartwig for their patience and continuous support.

Basic Notations

Let X be a set, and $A, B \subseteq X$ be subsets. We denote the complement of A by A^c, and set difference of A and B by $A \backslash B$. The set of real numbers is denoted by $\mathbb{N} := \{1, 2, \cdots\}$, the set of real numbers included zero by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the set of integers by $\mathbb{Z} := \{\cdots, -1, 0, 1, \cdots\}$, the set of real numbers by $\mathbb{R}$, the set of rational number by $\mathbb{Q}$, and the set of complex numbers by $\mathbb{C}$. A field is denoted by $\mathbb{F}$, which is in our case either $\mathbb{R}$ or $\mathbb{C}$. We denote the unit circle/1-dim. torus by $\mathbb{T}$.

Let X and Y be two sets. The notation $f : X \to Y$ stands for a function/mapping f with domain X and codomain Y. We denote the set of mappings between the sets X and Y as $\mathrm{Map}(X, Y)$. The image or range of the function f is the set $\mathrm{ran}(f) := \{f(x) : x \in X\}$. The direct image of a subset $S \subset X$ under f is denoted by $f(S) := \{f(t) : t \in S\}$. The inverse image of a subset $A \subseteq X$ is denoted by $f^{-1}(A) := \{y \in Y : f(y) \in A\}$. A function is said to be injective, or one-to-one, if for each $x \in X$, there is exactly one $y \in Y$, s.t. $f(x) = y$, or equivalently: if $f(a) = f(b)$ implies $a = b$. The function f is said to be surjective, or onto, if $f(X) = Y$. f is said to be bijective, if f is both injective and surjective. Let $S \subseteq X$, the restriction of f s.t. its domain is S, is denoted by $f|_S$. Given a subset A of a set X. The characteristic function of A is defined as a function $\chi_A : X \to \mathbb{R}$, for which $\chi_A(x) = 1$, $\forall x \in A$, and $\chi_A(x) = 0$, $\forall x \in A^c$ holds.

A set $\mathcal{I}$ is said to be an index set if all members of $\mathcal{I}$ label another set. Notice that an index set may also be an uncountable set. If it is not otherwise stated, $\mathcal{I}$ denotes an arbitrary index set.

Let be $\{X_i\}_{i \in \mathcal{I}}$ be a collection of sets indexed by an index set $\mathcal{I}$. The (cartesian) product of $\{X_i\}_{i \in \mathcal{I}}$ is defined as the set of functions:

$$\prod_{i \in \mathcal{I}} X_i := \{f : \mathcal{I} \to \bigcup_{i \in \mathcal{I}} X_i : \forall i \in \mathcal{I} : \; f(i) \in X_i\}.$$

If $X_n = X$, $\forall n \in \mathcal{I}$, we usually write X^I instead of $\prod_{n \in \mathcal{I}} X$. We shall often call X^I the power of X. For a countable index set $\mathcal{I}$, we call the elements of the power X^I of a set X sequence in X. Specifically, sequences in X has the form $\{x_n\}_{n \in \mathcal{I}}$, where $x_n \in X$, $\forall n \in \mathcal{I}$. If it is clear from context, we write simply $\{x_n\}_n$

1. Sampling and interpolation problem

Sampling theory concerns with the problem of reconstruction of a function from its values, taken at a countable set of points in its domain. This field is insofar important for electrical engineering applications, since sources of informations (e.g. which are objects of image processing - and audio processing applications) in the "real-world" are available in form of analog signals (which can be seen as functions whose domain is a connected subset of $\mathbb{R}^N$, where $N \in \mathbb{N}$) and most devices, in course of increasing digitalization, can only deal with countable data. Due to this limitation, it demands a transformation, which converts analog - to digital signals, of course, with a possibility to convert the transformed signal back, possibly without error.

The sampling and reconstruction problem can roughly be described as follows: Given a function f defined on some domain D. We look for expansions of f by the so-called sampling series:

$$f = \sum_{n \in \mathcal{I}} \tilde{f}(\lambda_n) S_n,$$

where $\mathcal{I}$ is an index set (necessarily countable), $\tilde{f}$ a function defined on D, which constitutes the preprocessed version of f ($\tilde{f}$=Tf, where T is some transformation), $\{\lambda_n\}_n$ a countable points in D, and $\{S_n\}_n$ a collection of functions, defined on D. In particular, we will especially consider sampling series, s.t. the collection of functions $\{S_n\}_n$ is basically the shifts of a suitable function ϕ by $\{\lambda_n\}_n$, specifically S_n, for which $S_n = \phi((\cdot) - \lambda_n)$, $\forall n \in \mathcal{I}$, holds.

A classical approach to the sampling and reconstruction problem for one-dimensional signals, i.e. signals which are defined in a domain of $\mathbb{R}$, was presented by Shannon in his paper "Communication in the presence of noise" in 1949 [52], which constitutes a milestone in the history of sampling theory. This work was based basically on Nyquist's considerations about transmission of finite sequence of numbers by means of trigonometric polynomials (see: [43]). Independently from it, Shannon's result was already given earlier in other similar forms by Raabe in 1939 [49], by Kotelnikow in 1933 [34], and by Whittaker in 1915 [57]. The Nyquist-Shannon Theorem ensures the reconstructibility of a signal by its samples, provided that the samples was taken by the rate at least twice of the bandwidth of the signal to be sampled. The correspondent reconstruction process can described by the formula (WRKS-formula):

$$f(x) = \sum_{k \in \mathbb{Z}} f(kT) \operatorname{sinc}(\frac{x}{T} - k), \tag{1.1}$$

With regard to above equation, a perfect reconstruction of f from the samples $\{f(kT)\}_{k \in \mathbb{Z}}$, taken uniformly on the time instances of integer multiples of T, can be obtained, if the representation f in the frequency domain is supported in the interval $[-\pi/T, \pi/T)$ i.e. $\hat{f}(\omega) = 0$, $\forall \omega \notin [-\pi/T, \pi/T)$. On the one hand, the signal obtained by this method is determined self-evidently by its samples $\{f(kT)\}_{k \in \mathbb{Z}}$, which gives an essential informations about the appearance of the signal object f. On the other hand, it is determined by the sequence $\{\operatorname{sinc}(\frac{x}{T} - k)\}_{k \in \mathbb{Z}}$, which declares the belonging of f to a certain signal space, which in this case: the space of square-integrable, band-limited signal, which is also known as the Paley-Wiener Space.

There exists successful efforts to extend above idea to other domains. For square-integrable band-limited functions on $\mathbb{R}^N$, Parzen provides in [44] (1956) a corresponding sampling formula for this case.

However, the possible shape of the band-limit is restricted to a rectangular symmetric around 0. The corresponding reconstruction formula can be basically seen as the WRKS-formula applied to each degrees of freedom:

$$f(x_1,\ldots,x_N) = \sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_N\in\mathbb{Z}} f(k_1T_1,\ldots,k_NT_N)\,\mathrm{sinc}(\frac{x_1}{T_1}-k_1)\cdots\mathrm{sinc}(\frac{x_N}{T_N}-k_N), \tag{1.2}$$

where the frequency occupation of the square-integrable signal f, for which above formula holds, is contained in the interval $[-\pi/T_1,\pi/T_1)\times\cdots\times[-\pi/T_N,\pi/T_N)$. However, by establishing a sampling theorem for signals band-limited to a hexagonal, Gaarder [16] asserts that the corresponding sampling points in non-rectangular cases are no more uniformly placed in $\mathbb{R}^N$, but still in some sense periodic. Furthermore, the keyword "periodicity" suggests to involve group theoretic view to this problem. Another result was made by Kluvanek [31] in 1965. In this work, he stated a sampling and reconstruction algorithm for square-integrable band-limited signal, whose domain is a so-called LCA group. In this work, we will concern ourselves with this sort of topological group. The theory of LCA group, as we shall see, provides a "nice" way to unify the signals which is of importance in the electrical engineering, viz. array signals, disrete-time signals, and continuous signals.

However, in many applications, e.g. multi-carrier systems, the frequency occupation of the signals of interests is contained in mutually disjoint subsets. The sampling methods proposes above is inefficient, since it yields a higher rate than the theoretical possible rate asserted by Landau and Beurling [35], which is related to the volume of the frequency occupation of the signal. There are some approaches to overcome this disbenefit. Sampling by the Sub-Nyquist method, e.g. [36], [23], and [54] yields an optimal choice. Moreover, there are some extension of the Sub-Nyquist to handle the case, in which the information of the frequency occupation of the signals to be sampled is partially available, e.g. see: [13], [61], [39], [40].

In particular, as we shall see later, the problem of sampling and reconstruction of finite-energy and band-limited signal is closely related to the problem of finding a (possibly redundant) representation system consisting of exponential functions for the space of square-integrable signals, supported on some subset of the considered domain. Recently, the problem of finding such a representation system has attracted some interests: The existence of Riesz bases for the Hilbert space $L^2(\Omega)$, where $\Omega\subseteq\mathbb{R}^N$ is a so called multi-tiling set, which is not necessarily a connected subset of $\mathbb{R}^N$, was proved in [17], using the theory of quasicrystals [37], [38]. A similar result was proven in [33] based on a more elementary approach from linear algebra. Agora et. al. generalize in [1] the result for general LCA groups. In this paper, we consider the construction of Riesz bases and frames together with the corresponding dual frames for $L^2(\mathbb{B})$ with domains. However, no attempt was made to derive the corresponding dual frames. Our approach here is more elementary and based on techniques used for shift-invariant spaces with support in $\mathbb{R}$ [46], allowing us to give the corresponding dual frame. Yielding from this result, we shall give a corresponding sampling and reconstruction scheme, which can basically seen as multi-channel sampling, involving linear preprocessor on each of the sampling branches. In special cases, the proposed sampling method coincides with WKRS-formula, and Sub-Nyquist sampling method. Furthermore, we shall state the result in general LCA group setting.

This thesis is organized as follows: In the first two chapters, some necessary mathematical fundamentals will be introduced. The *first chapter* is devoted to the theory of generating systems in Hilbert spaces. There, we shall introduce the notion of frames which constitute one fundamentals of our approach. The *second chapter* is devoted to the so-called locally compact group. In particular, we shall see, what does "frequency" means in connection to this sort of a topological group. The corresponding abstract form of Fourier transformation shall be given and studied there. The *third chapter* is devoted to the derivation of sampling theorem for square-integrable functions on LCA groups, whose spectrum is band-limited to a subset in an LCA groups. We will formalize the notions of not-necessarily uniform, but periodic, sampling points. We will see that the notion of "admissible" band-limit for periodic sampling, is determined by those points. In particular, we shall recapitulate, up to some modifications, the sampling and reconstruction algorithm proposed by Kluvanek in [31]. For convenient, we shall at the end of this chapter apply this algorithm to the case, that the signal domain is Euclidean, which yields a formula, similar to WKRS-formula. In the *fourth chapter*, we concern ourselves with the so-called phase retrieval problem. By taking the approach analogue to [60] and [48], and the sampling theorem introduced in chapter three, we shall see that the phase retrieval of infinite dimensional planar signal is possible, which extend the results given in [60] and [48]. The *fifth chapter* is devoted to the so-called frames of exponentials weighted by some collection of functions. We shall derive a necessary and sufficient conditions s.t. the collection of functions $\{\phi^{(m)}e_\lambda\}_{m,\lambda}$ forms a frame for the function space $L^2(\mathbb{B})$, where $\{\phi^{(m)}\}_m$ is a collection of functions defined on $\mathbb{B}$, $\{e_\lambda\}$ is an orthonormal basis for the function space $L^2(\mathbb{B})$, and $\mathbb{B}$ an appropriate subset of an LCA group. We shall show, that this sort of frames induces a sampling and reconstruction scheme. In the *sixth chapter*, Some explicit choices concerning to the so-called multi-coset sampling and Gabor system shall be considered in the sixth chapter. Finally, this thesis ends in the *seventh chapter* with a summary, discussions, and an outlook to future works.

2. Bases and frames

Later, we shall see, that the set of objects of our interests, viz. finite-energy band-limited signals, can be integrated into the theory of Hilbert space of square-integrable functions. Specifically, each signal f of our interests can be seen as an element of the Hilbert space $L^2(\mathcal{G})$, where $\mathcal{G}$ corresponds to the domain of f. Thus, the sampling and reconstruction problem of such signals can be translated appropriately into a mathematical problem, namely: find a collection of "primitive" elements, called generating system, of the Hilbert space $L^2(\mathcal{G})$ s.t. all $f \in L^2(\mathcal{G})$ can be expanded as an "infinite linear combination" of this collection.

The statements made in this chapter is taken up to some modifications from [47], [8], [56], and [22].

2.1. Basics on operator theory in Banach- and Hilbert spaces

In this section, if not otherwise stated, X and Y stand for normed spaces, $\mathcal{X}$ and $\mathcal{Y}$ for Banach spaces, and $\mathcal{H}$ and $\mathcal{K}$ for Hilbert spaces.

We start by introducing linear mapping between normed spaces. Thereby, we follow the following convention: An *operator* $\mathrm{V} : X \to Y$ is defined as a linear mapping between vector spaces, i.e. $\mathrm{V}(\alpha x + \beta y) = \alpha \mathrm{V}(x) + \beta \mathrm{V}(y)$, for $x, y \in \mathcal{X}$ and α, β scalary. If the image of this mapping is a field (in particular real or complex), so it is called a *functional*. If there is no possible confusions, $\mathrm{V}(x)$ shall be written as $\mathrm{V}x$. It is assumed, that definitions of pointwise - and uniform continuity of real-valued functions on the real line is known for the readers. By some modifications, one can easily extend this notions to mappings between normed spaces (use norm instead of modulus). The continuity of an operator between normed spaces can equivalently be expressed by means of convergent sequences: $\mathrm{V} : X \to Y$ is continuous, if for each sequence $\{x_n\}_n$, $x_n \to x$ w.r.t. the norm in X, implies $\mathrm{V}x_n \to \mathrm{V}x$ w.r.t. to the norm of Y.

One can easily prove that an operator is pointwise continuous if and only if it is uniformly continuous. Hence, if considering an operator between normed spaces, it is unnecessary to make a distinction between pointwise continuity and uniform continuity of an operator. Furthermore, to show that an operator is continuous, it is sufficient and necessary to show that it is continuous at 0, which clearly makes the analysis on operators w.r.t. this property easier.

It is common to define the norm of an operator by the following way: For an operator $\mathrm{V} : X \to Y$, define $\|\mathrm{V}\| := \inf\{M \geqslant 0 : \|\mathrm{V}x\| \leqslant M\|x\|, \quad \forall x \in \mathcal{X}\}$. To say roughly, the norm of an operator gives the maximal quantity of "gain", which each $x \in \mathcal{X}$ obtains, if it is mapped by V. It is not hard to see, that the norm of an operator $\mathrm{V} : X \to Y$ can also be written equivalently as:

$$\|\mathrm{V}\| = \sup_{x \neq 0} \frac{\|\mathrm{V}x\|}{\|x\|} = \sup_{\|x\|=1} \|\mathrm{V}x\| = \sup_{\|x\| \leqslant 1} \|\mathrm{V}x\|.$$

Note that, the following fundamental inequality concerning to operator norms holds:

$$\|\mathrm{V}x\| \leqslant \|\mathrm{V}\|\|x\|.$$

In case that there are possibilities for confusions, the norm is written more specifically by $\|\mathrm{V}\|_{X \to Y}$. An operator is said to be *bounded* if its norm is finite, else it is said to be unbounded. It is not hard to show, that an operator is continuous if and only if it is bounded. So, it seems likely, when speaking about operators, to interchange the terms "continuous" and "bounded". The set of bounded operators between normed spaces X and Y is denoted by $\mathcal{B}(X, Y)$, which is, as we have already seen, equivalent to the set of continuous operator between X and Y.

The following classes of bounded operators between Banach spaces shall be considered later:

- V is said to be isomorphic, if V is bijective and its inverse operator V^{-1} is continuous/bounded.
- V is said to be isometric if $\|\mathrm{V}x\| = \|x\|$, $\forall x \in \mathcal{X}$.
- V is said to be a isometric isomorphism if V is isometric and isomorphic.

If there exist an isometric isomorphic operator between $\mathcal{X}$ and $\mathcal{Y}$, then they both are said to be linear isometric isomorphic to each other, written: $\mathcal{X} \cong \mathcal{Y}$. We call an isometric isomorphic (resp. isometric) operator also linear isometric isomorphism (resp linear isometry). It is not hard to show, that linear isometry between Hilbert spaces can also alternatively be characterized as follows: $\mathrm{V} : \mathcal{H} \to \mathcal{K}$ is an isometry if and only if $\langle \mathrm{V}x, \mathrm{V}y \rangle = \langle x, y \rangle$, $\forall x, y \in \mathcal{X}$, i.e. V preserves the "correlation" between elements of $\mathcal{X}$. Sometimes, we call isometric isomorphic operator V between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ *unitary operator*. Also, we say V is a *unitary equivalence* between $\mathcal{H}$ and $\mathcal{K}$. From a topological point of view, isomorphic operator can also be seen as linear homeomorphism. As an application of the Open Mapping Theorem (of Banach spaces), it can be shown, that a bounded bijective operator between Banach spaces is indeed a linear homeomorphism. In particular, bounded bijective operator between Banach spaces preserves vector space structures, and simultaneously, an open - and a closed map.

To each bounded operator V between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, there corresponds a unique operator $\mathrm{V}^* : \mathcal{K} \to \mathcal{H}$ called the *adjoint operator* of V, which is determined by the equation:

$$\langle \mathrm{V}x, y \rangle_{\mathcal{K}} = \langle x, \mathrm{V}^* y \rangle_{\mathcal{H}}, \quad x \in \mathcal{H},\ y \in \mathcal{K}.$$

The existence and the uniqueness of adjoint of an operator is ensured by Riesz Representation Theorem. By some efforts, one can proof that the adjoint mapping $\mathcal{B}(\mathcal{H}, \mathcal{K}), \to \mathcal{B}(\mathcal{K} \to \mathcal{H})$, $\mathrm{V} \mapsto \mathrm{V}^*$, which assigns each element of the space of continuous mapping between Hilbert spaces the corresponding unique adjoint, is norm-preserving and conjugated linear, i.e. for $\mathrm{V}, \mathrm{V}_1, \mathrm{V}_2 \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, and $a, b \in \mathbb{F}$, it holds:

- $\|\mathrm{V}^*\| = \|\mathrm{V}\|$,
- $(a\mathrm{V}_1 + b\mathrm{V}_2)^* = \overline{a}\mathrm{V}_1^* + \overline{b}\mathrm{V}_2^*$.

Furthermore, the following statements concerning to the range - and the kernel of an $\mathrm{V} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and its adjoint V^*, are equivalent:

- $\mathrm{ran}(\mathrm{V})$ is closed
- $\mathrm{ran}(\mathrm{V}) = \ker(\mathrm{V}^*)^{\perp}$
- $\mathrm{ran}(\mathrm{V}^*)$ is closed
- $\mathrm{ran}(\mathrm{V}^*) = \ker(\mathrm{V})^{\perp}$

To define the adjoint of a not necessarily bounded operator, one needs some efforts:

Definition 2.1 (Adjoint of a Not Necessarily Bounded Operator). *Let $\mathrm{V} : \mathcal{K} \to \mathcal{H}$ be an operator between Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$. An operator $S : \mathcal{H} \supset \mathit{Dom}\, S \to \mathcal{K}$ is called the formal adjoint of* V*, if the following equality is fulfilled:*

$$\langle y, \mathrm{V}x \rangle_{\mathcal{H}} = \langle Sy, x \rangle_{\mathcal{K}}, \quad \forall x \in \mathcal{K},\ y \in \mathit{Dom}\, S,$$

where $\mathrm{Dom}(S)$ *is the subset of* $\mathcal{H}$*, containing points* y *for which* $x \mapsto \langle y, \mathrm{V}x\rangle_{\mathcal{H}}$ *is a continuous/bounded functional. The operator* S *with* $\mathrm{Dom}(S)$ *maximal under the partial order of set containment in* $\mathcal{H}$ *is called the adjoint of* V *and denoted by* V^**.*

We say, an operator $\mathrm{V} : \mathcal{X} \to \mathcal{Y}$ is densely defined, if its domain is dense in $\mathcal{X}$. Later, we need the following statement:

Proposition 2.1. *Let* $\mathrm{V} : \mathcal{K} \to \mathcal{H}$*,* $\tilde{\mathrm{V}} : \mathcal{H} \to \tilde{\mathcal{H}}$ *be dense defined operators mapping between Hilbert spaces. Then:*

(a) V *is bounded if and only if* $\mathrm{V}^* \in \mathcal{B}(\mathcal{H}, \mathcal{K})$*. In this case, it holds:* $\|\mathrm{V}\| = \|\mathrm{V}^*\|$.

(b) If $\tilde{\mathrm{V}} \in \mathcal{B}(\mathcal{H}, \tilde{\mathcal{H}})$*, then it holds* $(\tilde{\mathrm{V}}\mathrm{V})^* = \mathrm{V}^*\tilde{\mathrm{V}}^*$.

The following easy statements might also be helpful for later approach:

Lemma 2.2. *Let* $\mathcal{X}$*,* $\mathcal{Y}$ *be Banach spaces,* $A \subseteq \mathcal{X}$ *a closed subspace. The restriction* $\mathrm{T}|_A : A \to \mathrm{T}(A)$ *of an (isometric) isomorphism* $\mathrm{T} : \mathcal{X} \to \mathcal{Y}$ *is again an (isometric) isomorphism.*

Proof. Since A a closed subspace of the Banach space $\mathcal{X}$, it follows that A, equipped with the structure inherited from $\mathcal{X}$, is itself a Banach space. The restriction $\mathrm{T}|_A : A \to \mathrm{T}(A)$ of T to the closed subspace A remains clearly linear (provided that $T(A)$ is equipped with vector space structure inherited from $\mathcal{Y}$), injective, and surjective, and bounded. Furthermore, from linearity of T, and since A is a subspace, it follows immediately that $\mathrm{T}(A)$ is a subspace of Y. Summarily, $T|_A$ is a bijective bounded operator from A to $\mathrm{T}(A)$, and accordingly a linear homeomorphism between both spaces, thence it is clear that $\mathrm{T}(A)$ must also be closed. Equipping $\mathrm{T}(A)$ with the Banach space structure inherited from $\mathcal{Y}$ is also itself a Banach space. Accordingly, $\mathrm{T}|_A$ is an isomorphism between A to $\mathrm{T}(A)$. Let T be in addition isometric. The fact, that $\mathrm{T}|_A : A \to \mathrm{T}(A)$ is also isometric, is obvious. ∎

Lemma 2.3. *Let* $\mathcal{X}_1$*,* $\mathcal{X}_2$*, and* $\mathcal{X}_3$ *be Hilbert spaces. Further, assume that* $\mathcal{X}_2$ *and* $\mathcal{X}_3$ *are isometric isomorphic to each other, by the isometric isomorphism* $\mathrm{T} : \mathcal{X}_2 \to \mathcal{X}_3$*. Given an operator* $\mathrm{V} : \mathcal{X}_1 \to \mathcal{X}_2$*. If* $\mathrm{T} \circ \mathrm{V} : \mathcal{X}_1 \to \mathcal{X}_3$ *is bounded and surjective, then* V *is also bounded and surjective.*

Proof. Since T is assumed to be an isometric isomorphism, T is clearly bounded. Hence, it holds: $(\mathrm{T} \circ \mathrm{V})^* = \mathrm{V}^* \circ \mathrm{T}^*$. It follows from the assumption that $\mathrm{T} \circ \mathrm{V}$ is bounded and surjective, that $(\mathrm{T} \circ \mathrm{V})^*$, and respectively also $\mathrm{V}^* \circ \mathrm{T}^*$ is bounded and bounded below. Since T is isometric isomorphic, and accordingly T^* is also isometric isomorphic, it follows that V^* is bounded and bounded below, which shows the desired statement. ∎

Proposition 2.4. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be Banach spaces, and* D *a dense subset of* $\mathcal{X}$*. Given an operator* $\mathrm{T} \in \mathcal{B}(D, \mathcal{Y})$*. Then there exists a unique continuous operator* $\hat{\mathrm{T}} : \mathcal{X} \to \mathcal{Y}$*, for which* $\hat{\mathrm{T}}\big|_D = \mathrm{T}$*, and* $\|\hat{\mathrm{T}}\| = \|\mathrm{T}\|$ *holds. The corresponding operator* $\hat{T}$*, can be defined by the following process: For each convergent sequence* $\{x_n\}_{n\in\mathbb{N}}$*, converges to an* $x \in \mathcal{X}$*, define* $\hat{T}x := \lim_n Tx_n$*.*

2.1.1. Direct Sum of Operators

Let $\{\mathcal{B}_k\}_{k\in I}$ be a countable family of Banach space, we write $\bigoplus_{k\in\mathcal{I}} \mathcal{B}_k$ for the direct sum of the elements of that family, i.e.

$$\bigoplus_k \mathcal{B}_k := \{\mathbf{f} := \{f_k\}_{k\in\mathcal{I}} \in \prod_{k\in\mathcal{I}} \mathcal{B}_k : \ f_k \neq 0, \ \forall k \in I \subseteq \mathcal{I}, \ I \text{ finite}\},$$

where $\prod$ denotes the usual cartesian product. Equipping $\bigoplus_{k\in\mathcal{I}}\mathcal{B}_k$ with the usual addition and scalar multiplication canonical to the product structure, it is not hard to see that $\bigoplus_{k\in\mathcal{I}}\mathcal{B}_k$ becomes a vector space. Furthermore, one can show that $\oplus_k\mathcal{B}_k$ equipped with the norm:

$$\|f\| := \sum_{k\in\mathcal{I}}\|f_k\|_{\mathcal{H}_k}$$

is a Banach space. If $\mathcal{B}_k = \mathcal{B}$, where $\mathcal{B}$ is a Banach space, we write the direct sum of $\{\mathcal{B}_k\}_{k\in\mathcal{I}}$ simply by $\mathcal{B}^{\oplus\mathcal{I}}$. In case that $\mathcal{I}$ is finite, and $\mathcal{I} = [K]$, we write simply $\mathcal{B}^{\oplus K}$ instead of $\mathcal{B}^{\oplus[K]}$. Consider the direct sum $\mathcal{B} := \oplus_{k\in\mathcal{I}}\mathcal{B}_k$ of Banach spaces. By the notation $x\boldsymbol{\delta}_k$, where $k \in [K]$ and $x \in \mathcal{B}_k$, it is meant, the sequence/vector in $\mathcal{B}$, which is x on the k^{th}-element/entry, and zero elsewhere.

Given a direct sum $\bigoplus_{k\in\mathcal{I}}\mathcal{H}_k$ of Hilbert spaces. We can canonically define an inner product on $\bigoplus_{k\in\mathcal{I}}\mathcal{H}_k$ by:

$$\langle \mathbf{f}, \mathbf{g}\rangle_{\bigoplus_{k\in\mathcal{I}}\mathcal{H}_k} := \sum_{k\in\mathcal{I}}\langle f_k, g_k\rangle, \quad \forall f, g \in \bigoplus_{k\in\mathcal{I}}\mathcal{H}_k.$$

It is not hard to show, that $\bigoplus_{k\in\mathcal{I}}\mathcal{H}_k$ equipped by this structure is a Hilbert space. The direct sum of Hilbert spaces, which we shall mostly consider is the direct sum of Lebesgue spaces, viz. the direct sum of the form $\bigoplus_{k\in\mathcal{I}}L^2(\Omega_k)$, $\{\Omega_k\}_{k\in\mathcal{I}}$ is a collection of measure spaces. Let $\mathcal{G}$ be a measure space (later, $\mathcal{G}$ will be mostly an LCA group, equipped with the corresponding Borel σ-algebra, and Haar measure), and $\Omega \subseteq \mathcal{G}$ a measureable subset. Given an $\mathbf{f} := (f_1, \ldots, f_K) \in L^2(\Omega)^{\oplus K}$. Since the domains of each $\{f_k\}_{k\in[K]}$ coincides, it stands clear to treat $\mathbf{f}$ as a vector-valued function $\mathbf{f} : \Omega \to \mathbb{C}^K$, $x \mapsto (f_1(x), \ldots, f_K(x))$.

Given a collection of operators, each acts on a Banach space. One can canonically define the following class operator:

Definition 2.2 (Direct Sum Operator). *Let $\{B_k\}_{k\in\mathcal{I}}$ and $\{B_k'\}_{k\in\mathcal{I}}$ be countable collections of Banach spaces. Write the direct sum of $\{B_k\}_{k\in\mathcal{I}}$ by $\mathcal{B}$, and the direct sum of $\{B_k'\}_{k\in\mathcal{I}}$by $\mathcal{B}'$.*

Let $T_k : B_k \to B_k'$, $k \in I$, be a collection of bounded operators. The direct sum operator $T := \oplus_{k\in\mathcal{I}}T_k : \mathcal{B} \mapsto \mathcal{B}'$ is defined by:

$$T(\bigoplus_{k\in\mathcal{I}}x_k) := \bigoplus_{k\in\mathcal{I}}T_kx_k.$$

The following lemma, concerning to invertibility of direct sum operator, is not hard to establish:

Lemma 2.5. *Given a direct sum operator $T = \oplus_{k\in\mathcal{I}}T_k : \mathcal{B} \to \mathcal{B}'$ mapping between direct sum Banach spaces $\mathcal{B} = \oplus_{k\in\mathcal{I}}B_k$ and $\mathcal{B}' = \oplus_{k\in\mathcal{I}}B_k'$. Then T is invertible, if and only if each T_i, $i \in \mathcal{I}$, is invertible. Correspondingly, the inverse of T is given by the direct sum operator $T^{-1} := \oplus_{k\in\mathcal{I}}T_k^{-1}$.*

2.2. Some Classes of Generating systems in Hilbert spaces

First, we recall some facts about sequences and series. Consider a normed vector space $\mathcal{X}$ and let $\{x_n\}$ be a sequence in $\mathcal{X}$. $\{x_n\}$ is said to converge to $x \in \mathcal{X}$ if: $\lim_{n\to\infty}\|x - x_n\| = 0$, and $\{x_n\}$ is said to be a cauchy sequence if for all $\epsilon > 0$ the exist a number $N \in \mathcal{I}$ s.t.: $\|x_n - x_m\| < \epsilon, \quad \forall n, m \geqslant N$. Convergent sequences are automatically Cauchy sequences. The converse is true, in the case that the considered space is in addition a Banach space (which is exactly the definition of this space). A sequence $\{x_n\}$ in a normed vector space is said to be bounded below (bounded above) if $\inf_n\|x_n\| > 0$ ($\sup_n\|x_n\| < \infty$), respectively. It is not hard to see that all Cauchy sequences in a normed space is bounded above. Let $\{x_n\}_{n\in\mathcal{I}}$ be a sequence in a Banach space $\mathcal{B}$ and consider the series in the form $\sum_{n\in\mathcal{I}}x_n$. $\sum_{n\in\mathcal{I}}x_n$ is said to converge unconditionally if $\sum_{n\in\mathcal{I}}x_{\sigma(n)}$ converges, with $\sigma(\cdot)$ denotes an arbitrary permutation on $\mathcal{I}$. Otherwise, $\{x_n\}$ is said to converge conditionally. $\sum_{n\in\mathcal{I}}x_n$ is said to converge absolutely if $\sum_{n\in\mathcal{I}}\|x_n\|$ converges.

In this section, we consider ourself with generating systems in Hilbert spaces $\mathcal{H}$, i.e. collection A of countable elements in that space which allows a representation of each element of $\mathcal{H}$ as an " infinite linear combination" of elements of A, where the corresponding representation has to be seen as limiting process. More specific, we consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ s.t. any element x of $\mathcal{H}$ can be written, not necessarily in a unique way, as $x = \sum_{n=1}^{\infty} c_n x_n$, for some scalars $\{c_n\}_{n\in\mathbb{N}}$, where the equality has to be understand as: $x = \lim_{n\to\infty} \sum_{k=1}^{n} c_k x_k$, with the convergence is w.r.t. the norm of $\mathcal{H}$.

Let $\mathcal{X}$ be a vector space over a $\mathbb{F}$ (which is in our case either complex - or real field), and $\{x_n\}_{n=1}^{\infty}$ be a countable collection of elements of $\mathcal{X}$. The expression $x = \sum_{n=1}^{\infty} c_n x_n$, with $c_n \in \mathbb{F}\ \forall n$, is called expansion of x w.r.t. $\{x_n\}_{n\in\mathbb{N}}$. In the following subsection, we give some discussions on generating systems in Hilbert spaces $\mathcal{X}$, which provides a unique expansion of each element of $\mathcal{X}$.

2.2.1. Bases for Hilbert Spaces

On a Banach space, one can provide a generating system as follows:

Definition 2.3 (Basis). *Let $\mathcal{X}$ be a Banach space and $\{x_n\} \subset \mathcal{X}$ countable. $\{x_n\}$ is said to be a basis of $\mathcal{X}$ if the expansion of every element of $\mathcal{X}$ w.r.t. $\{x_n\}$ is unique. The corresponding expansion is called basis expansion.*

Let $\{x_n\}$ be a basis of $\mathcal{X}$. $\{x_n\}$ is said to be unconditional/absolutely convergence if the basis expansion of each element of $\mathcal{X}$ converges unconditionally/absolutely, respectively.

The following easy but useful lemma asserts the invariance of bases under linear homeomorphism.

Lemma 2.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $\mathrm{V} : \mathcal{X} \to \mathcal{Y}$ be a linear homeomorphism. If $\{x_n\}$ is a basis in $\mathcal{X}$, then $\{\mathrm{V}x_n\}$ is a basis in $\mathcal{Y}$. In particular, let $f \in \mathcal{X}$, and consider the representation $f = \sum_n c_n x_n$ by means of the basis $\{x_n\}_n$, then it holds $\sum_n c_n \mathrm{V}x_n$ converge w.r.t. the norm of $\mathcal{Y}$ to $\tilde{f}$, where $\tilde{f} = \mathrm{V}f$.*

Proof. Consider an arbitrary $y \in \mathcal{Y}$. There corresponds an $x \in \mathcal{X}$ with basis expansion $\sum_n c_n x_n$, s.t. $x = \mathrm{V}^{-1}y$. Applying V to that basis expansion and by linearity and continuity of V, one obtains $y = \sum_n c_n \mathrm{V}x_n$. Suppose that there is another representation $y = \sum_n b_n \mathrm{V}x_n$. Applying V^{-1} to this representation and by continuity and linearity of this mapping, one gets $\sum_n b_n x_n$. Since basis expansion is always unique, it yields $c_n = b_n$, $\forall n$. Hence, the statements hold. ∎

Now, it can be said that two bases $\{x_n\}$ and $\{y_n\}$ for Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively, possessing a linear homeomorphism $\mathrm{V} : \mathcal{X} \to \mathcal{Y}$ is V-*equivalent* or simply: *equivalent* (written $\{x_n\} \approx^{\mathrm{V}} \{y_n\}$ or simply: $\{x_n\} \approx \{y_n\}$) if $y_n = \mathrm{V}x_n$. The following theorem states that convergence properties of a basis is preserved by linear homeomorphism:

Theorem 2.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $\{x_n\}$ be a basis in $\mathcal{X}$, and $\{y_n\}$ be a basis in $\mathcal{Y}$. Then the following statements are equivalent:*

1. *$\{x_n\} \approx \{y_n\}$.*
2. *$\sum c_n x_n$ converges (-/unconditionally/absolutely) if and only if the infinite sum $\sum c_n y_n$ converges (-/unconditionally/absolutely).*

Proof. From 1. to 2.:

Suppose $\{x_n\} \approx_{\mathrm{V}} \{y_n\}$ with a linear homeomorphism V and take a convergence sum $\sum c_n x_n$. Applying V to this sum and by linearity of V, it yields $\sum c_n \mathrm{V}x_n$ which converges since V is continuous. For the other way, note that $\approx$ is symmetric. So one can show that analogously. ∎

Especially, we are interested in the following classes of bases for Hilbert spaces:

Definition 2.4 (Orthonormal Basis). *Let $\mathcal{H}$ be a Hilbert space, and $\{e_n\}$ be a countable sequence in $\mathcal{H}$. $\{e_n\}$ is said to be an orthonormal basis for $\mathcal{H}$ if $\{e_n\}$ is a basis for $\mathcal{H}$ and $\{e_n\}$ is an orthonormal system in $\mathcal{H}$, i.e. $\langle e_k, e_l\rangle = \delta_{k,l}$.*

Orthonormal bases constitute an important class of bases of a Hilbert space. As a consequence of the fact, that bases are preserved by linear homeomorphisms, and the fact that unitary operators preserve correlations, one obtains immediately the following statement, which is simply a modification of lemma 2.6:

Lemma 2.8. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be (separable) Hilbert spaces. Suppose that there exists a isometric isomorphism $\mathrm{V} : \mathcal{H}_1 \to \mathcal{H}_2$. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an orthonormal basis of $\mathcal{H}_1$. Then $\{\mathrm{V}e_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis of $\mathcal{H}_2$. In particular, let $x = \sum_\lambda c_\lambda e_\lambda$ be an expansion of an $x \in \mathcal{H}_1$ by means of $\{e_\lambda\}$. Then $\sum_\lambda c_\lambda \mathrm{V} e_\lambda$ is an expansion of $\mathrm{V}x$.*

In case that each Hilbert space $\mathcal{H}_i$ in the finite collection $\{\mathcal{H}_i\}$ is separable, one can easily construct an orthonormal basis for the direct sum of this collection:

Proposition 2.9. *Let $\{\mathcal{H}_j\}_{j\in[n]}$ be a finite collection of separable Hilbert spaces. For each $j \in [n]$, denote the ONB for $\mathcal{H}_i$ by $\{e_\lambda^{(j)}\}_{\lambda\in\Lambda^{(j)}}$, where $\Lambda^{(j)}$ is a countable index set. Then, an orthonormal basis for $\oplus_{j\in[n]}\mathcal{H}_j$ is the union $\bigcup_{j\in[n]}\{e_\lambda^{(j)}\boldsymbol{\delta}_j\}_{\lambda\in\Lambda}$.*

From an orthonormal basis one can construct another class of bases for Hilbert spaces, called Riesz bases:

Definition 2.5 (Riesz Basis). *Let $\mathcal{H}$ be a Hilbert space, and $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$. A countable sequence $\{\phi_n\}$ is said to be a Riesz basis if:*

$$\phi_n = \mathrm{V}e_n, \quad \forall n,$$

for a $\mathrm{V} : \mathcal{H} \to \mathcal{H}$ bounded bijective operator. A sequence $\{\phi_n\}$ is said to be a Riesz sequence, if it is a Riesz basis for its closed span.

It is obvious that orthonormal bases are also Riesz bases. It is well-known, that Riesz basis is a bounded unconditional basis for a Hilbert space. Furthermore, there is an equivalent characterization of a Riesz basis: $\{x_n\}_n$ is a Riesz basis for a Hilbert space $\mathcal{H}$, if and only if there exists constant $A, B > 0$ s.t.:

$$A\sum_n |c_n|^2 \leqslant \|\sum_n c_n x_n\| \leqslant B\sum_n |c_n|^2, \quad \forall\{c_n\} \in l^2. \tag{2.1}$$

It can be shown, that to check whether (2.1) holds for some positive constant A and B, it is sufficient and necessary to check (2.1) only for all sequences, with only finitely many non-zero elements.

2.2.2. Frames for Hilbert Spaces

Let be $\{x_n\}_n$ a countable collection of elements of a Hilbert space $\mathcal{H}$. We say $\{x_n\}_n$ is said to be a Bessel sequence if: $\sum_n |\langle x, x_n\rangle|^2 < \infty, \quad \forall x \in \mathcal{H}$. Suggested by previous definition, one can define the so called *analysis operator* corresponding to a sequence $\{x_n\}_n$ mapping from a Hilbert space $\mathcal{H}$ to a sequence space, by $\mathrm{C}x = \{\langle x, x_n\rangle\}_n$, $x \in \mathcal{H}$. One immediately see, that the analysis operator corresponding to a Bessel sequence is a bounded operator from a Hilbert space into l^2. Furthermore, expansions w.r.t. a Bessel sequence, provided that the corresponding coefficients is quadratic summable, converges. So, corresponding to a Bessel sequence $\{x_n\}_n$ in a Hilbert space $\mathcal{H}$, one can define another bounded mapping from l^2 into $\mathcal{H}$, by $\mathrm{R}\{c_n\}_n = \sum_n c_n x_n, \quad \forall\{c_n\} \in l^2$, which is also known as the *synthesis operator* corresponding to the Bessel sequence $\{x_n\}_n$. Furthermore, one can show, that latter sum converges unconditionally. Notice that R is the adjoint operator of C. By composition of synthesis - and analysis operator of a Bessel sequence $\{x_n\}$ in a Hilbert space $\mathcal{H}$, one obtains a well-defined and bounded operator called frame operator $\mathrm{S} = \mathrm{RC}$.

Now we give in the following the notion of generating systems in a Hilbert space, which provides a not-necessarily unique expansion of the elements of the considered space:

Definition 2.6. *Let $\mathcal{H}$ be a separable Hilbert space, and $\{\phi_\lambda\} \subset \mathcal{H}$ a countable family. $\{\phi_\lambda\}$ is said to be a frame, if there exist $A, B > 0$ s.t.:*

$$A\|x\|^2 \leqslant \sum_\lambda |\langle x, \phi_\lambda\rangle|^2 \leqslant B\|x\|^2, \quad \forall x \in \mathcal{H}. \tag{2.2}$$

A is called lower frame bound and B upper frame bound. For ease of notations, we write sometimes $\sum_\lambda |\langle x, \phi_\lambda\rangle|^2 \approx \|x\|^2$ instead of (2.2).

We call a frame is *tight* if $A = B$. Furthermore, a frame is said to be *exact*, if it ceases to be a frame when an element of this collection is removed. One can show, that a Riesz basis is basically an exact frame, and an ONB is a tight exact frame.

Some statements of particular interests about analysis -, synthesis -, and frame operator corresponding to a frame $\{\phi_\lambda\}$ in a Hilbert space $\mathcal{H}$ can be given. One can show that the analysis operator corresponding to $\{\phi_\lambda\}$ is a bounded operator with closed range. Furthermore, the frame operator S corresponding to $\{\phi_\lambda\}$ is a positive invertible operator. Involving a left inverse of S, the latter fact asserts a way to reconstruct each element $x \in \mathcal{H}$ by means of $\{\phi_\lambda\}$. So, given a frame for a Hilbert space. By means of the so called canonical dual frame one obtain a reconstruction formula for each elements of this Hilbert space:

Theorem 2.10 (Reconstruction formula - Canonical Frame Expansion). *Given a frame $\{\phi_n\}$ for a Hilbert space $\mathcal{H}$ with frame bounds $A, B > 0$. The sequence $\{\mathrm{S}^{-1}\phi_n\}$ is a frame with frame bounds $B^{-1}, A^{-1} > 0$, which is called the canonical dual frame. Every $x \in \mathcal{H}$ can be expanded in the following way:*

$$x = \sum \langle x, \mathrm{S}^{-1}\phi_n\rangle \phi_n, \quad \text{and } x = \sum \langle x, \phi_n\rangle \mathrm{S}^{-1}\phi_n.$$

Furthermore, both sums converge unconditionally.

As same as bases, frames constitute a class of generating system in a Hilbert space, which is preserved by linear homeomorphism:

Lemma 2.11. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let $\{\phi_n\}_n$ be a frame for a Hilbert space $\mathcal{H}$, and $\mathrm{V} : \mathcal{H} \to \mathcal{K}$ be a linear homeomorphism. Then $\{\mathrm{V}\phi_n\}_n$ is a frame for $\mathcal{K}$. If A and B are frame bounds of $\{\phi_n\}_n$, then $\{\mathrm{V}\phi_n\}_n$ has the frame bounds $A\|\mathrm{V}^{-1}\|$ and $B\|\mathrm{V}\|$. If S is the frame operator for $\{\phi_n\}_n$, then VSV^* is the frame operator of $\{\phi_n\}_n$. Furthermore, $\{\phi_n\}$ is exact, if and only if $\{\mathrm{V}\phi_n\}_n$ is exact.*

For later approaches, we need the following statements:

Lemma 2.12. *Let $\mathcal{K}$ and $\mathcal{H}$ be (separable) Hilbert spaces, $\{e^{\mathcal{K}}_\lambda\}_\lambda$ be an orthonormal basis in $\mathcal{K}$ and $\mathrm{V} : \mathcal{K} \mapsto \mathcal{H}$ an operator. Then:*

1. *V is bounded and surjective if and only if $\{\mathrm{V}e^{\mathcal{K}}_\lambda\}$ is a frame for $\mathcal{H}$.*
2. *Let $\{\mathrm{V}e^{\mathcal{K}}_\lambda\}$ be a frame for $\mathcal{H}$, and let W be a bounded left inverse of V^*. Then $\{\mathrm{W}e^{\mathcal{K}}_\lambda\}$ is a dual frame of $\{\mathrm{V}e^{\mathcal{K}}_\lambda\}$.*
3. *Suppose additionally that $\mathcal{K}$ and $\mathcal{H}$ are isometric isomorphic to each other. V is bijective if and only if $\{\mathrm{V}e^{\mathcal{K}}_\lambda\}$ is a Riesz basis for $\mathcal{H}$.*

4. *Let* $\{\mathrm{V}e_\lambda^{\mathcal{K}}\}$ *is Riesz basis for* $\mathcal{H}$*. Then* $\{\mathrm{V}^{*,-1}e_\lambda^{\mathcal{K}}\}$ *is a dual Riesz basis of* $\{\mathrm{V}e_\lambda^{\mathcal{K}}\}$*.*

Proof. For the right implication of the statement 1.: From the assumption that V is bounded, it follows that $\mathrm{V}^* : \mathcal{H} \to \mathcal{K}$ is uniquely determined, and that they possess the same norm. In particular, V^* is also bounded. Furthermore, since V is assumed to be surjective, it is elementary to show that V^* is an injective map from $\mathcal{H}$ onto the closed set $\mathcal{R}(\mathrm{V}^*)$. So, by the open mapping theorem, it follows that V^* is bounded below. So, collecting the previous observations, it yields, that there exist $A, B > 0$ s.t.:

$$A\|x\|^2 \leqslant \|\mathrm{V}^*x\|^2 \leqslant B\|x\|^2, \quad \forall x \in \mathcal{H}. \tag{2.3}$$

Let $x \in \mathcal{H}$ and $y \in \mathcal{K}$ be the image of x under V^*. By means of the orthonormal basis $\{e_\lambda^{\mathcal{K}}\}_\lambda$ for $\mathcal{K}$, one can write the expansion of y as follows: $y = \sum_\lambda \langle y, e_\lambda^{\mathcal{K}}\rangle e_\lambda^{\mathcal{K}}$. By computation, the previous expansion can be written as: $y = \sum_\lambda \langle x, \phi_\lambda\rangle e_\lambda^{\mathcal{K}}$, where $\phi_\lambda := \mathrm{V}e_\lambda^{\mathcal{K}}$, $\forall\lambda$. So by the Parseval identity:

$$\|y\|^2 = \|\mathrm{V}^*x\|^2 = \sum_\lambda |\langle x, \phi_\lambda\rangle|^2. \tag{2.4}$$

Finally, setting this equality to (2.3), it is shown that $\{\phi_\lambda\}_\lambda$ is a frame for $\mathcal{H}$ with frame bounds A, B.

For the left implication of the first equivalence: Suppose that $\{\mathrm{V}e_\lambda^{\mathcal{K}}\}_\lambda$ is a frame for $\mathcal{H}$. It follows that

$$\sum_\lambda |\langle x, \mathrm{V}e_\lambda^{\mathcal{K}}\rangle|^2 \asymp \|x\|^2, \quad \forall x \in \mathcal{H}. \tag{2.5}$$

In particular, there exists a finite $B > 0$, s.t. for all $x \in \mathcal{H}$, it holds that $\sum_\lambda |\langle x, \mathrm{V}e_\lambda^{\mathcal{K}}\rangle|^2 \leqslant B\|x\|^2$, $\forall \lambda \in \Lambda$. Applying previous observation, and taking care the fact that $\{e_\lambda^{\mathcal{K}}\}_\lambda$ is an ONB for $\mathcal{K}$, it follows by some elementary computations [1] , that the functional $y \mapsto |\langle x, \mathrm{V}y\rangle_{\mathcal{H}}|$ is continuous for all $\forall x \in \mathcal{H}$. Hence $\mathrm{V}^* : \mathcal{K} \to \mathcal{H}$ exists. Finally, by rewriting (2.5), it yields:

$$\sum_\lambda |\langle x, \mathrm{V}e_\Lambda^{\mathcal{K}}\rangle|^2 = \sum_\lambda |\langle \mathrm{V}^*x, e_\Lambda^{\mathcal{K}}\rangle|^2 = \|\mathrm{V}^*x\| \asymp \|x\|^2, \quad \forall x \in \mathcal{H},$$

where the second equality follows from Parseval's Theorem. Hence, $\{\mathrm{V}e_\lambda^{\mathcal{K}}\}_\lambda$ is a frame for $\mathcal{H}$.

Clearly, the left inverse $\mathrm{W} : \mathcal{K} \to \mathcal{H}$ of V^* is linear, bounded, and surjective. Let $x \in \mathcal{H}$ be arbitrary. The image of x under V^* can be written as the orthonormal expansion: $\mathrm{V}^*x = \sum \langle x, \tilde{\phi}_\lambda\rangle e_\lambda^{\mathcal{K}}$, where $\tilde{\phi}_\lambda := \mathrm{V}e_\lambda^{\mathcal{K}}$. Applying W to the previous expansion, it yields: $x = \mathrm{W}\mathrm{V}^*x = \sum \langle x, \tilde{\phi}_\lambda\rangle \phi_\lambda$, which shows the desired statement.

For the second statement: Let $\mathcal{K}$ and $\mathcal{H}$ be isometric isomorphic to each other by the operator $\mathrm{T} : \mathcal{K} \to \mathcal{H}$. Define the operator $\mathrm{V}' := \mathrm{V} \circ \mathrm{T}^* : \mathcal{H} \to \mathcal{H}$, where $\mathrm{V} : \mathcal{K} \to \mathcal{H}$ is an operator. If V is bounded and bijective, it follows immediately that V', which is composition of bounded and bijective operators, is also bounded and bijective. Conversely, by noticing that $\mathrm{V} = \mathrm{V}' \circ \mathrm{T}$, and arguments analogously to the previous observation, it follows that if V is bounded and bijective, so also V' is bounded and bijective. Summarily V is bounded and bijective if and only if V' is bounded and bijective. From the fact that for all $\lambda \in \Lambda$, $\mathrm{V}e_\lambda^{\mathcal{K}} = (\mathrm{V}' \circ \mathrm{T})e_\lambda^{\mathcal{K}}$, that an orthonormal basis is preserved by isometric isomorphism the equivalence is shown.

Now, let $\{\mathrm{V}e_\lambda^{\mathcal{K}}\}_\lambda$ be a Riesz basis. Then V is a bounded bijective operator. By noticing that V^* is

[1] Let $y \in \mathcal{K}$. Expand: $y = \sum_{\lambda\in\Lambda} c_\lambda e_\lambda^{\mathcal{K}}$ for a unique $\{c_\lambda\} \in l^2$. $\forall x \in \mathcal{H}$, it follows:

$$|\langle x, \mathrm{V}y\rangle|^2 \leqslant \sum_{\lambda\in\Lambda} |c_\lambda|^2 |\langle x, \mathrm{V}e_\Lambda^{\mathcal{K}}\rangle|^2 \leqslant B\|x\|^2 \sum_{\lambda\in\Lambda} |c_\lambda|^2 \leqslant B'\|x\|^2,$$

where B, B' are finite non-negative scalars. The second inequality follows from the linearity of V, from some properties of inner product, from some elementary inequalities, from the fact $|\overline{a}| = |a|$, and the third inequality follows from the fact $\sum_\lambda |\langle x, \mathrm{V}e_\Lambda^{\mathcal{K}}\rangle|^2 \leqslant B\|x\|^2$ with $B > 0$ is a finite scalar, and finally the fourth inequality follows from the assumption that $\{c_\lambda\} \in l^2$.

bijective and bounded, and accordingly its left inverse $(\mathrm{V}^*)^{-1}$ (which is also its right inverse) is unique and bounded, and finally by computations analouge to the proof of of the second statement, the fourth statement follows. Since V and V^* have the same norm, it clearly follows that the frame bounds of the Riesz basis $\{\mathrm{V}e_\lambda^{\mathcal{K}}\}_\lambda$ coincides with the bounds of the operators V and V^*. ∎

We call the frame $\{\phi_\lambda\}_{\lambda\in\Lambda}$ for a Hilbert space $\mathcal{H}$, whose members are each the image of an orthohormal basis $\{e_\lambda\}_{\lambda\in\Lambda}$ under a bounded surjective operator $\mathrm{V}:\mathcal{K}\to\mathcal{H}$, the frame generated by V.

Remark 2.13. *Given an operator* $\mathrm{V}:\mathcal{K}\to\mathcal{H}$. *Suppose that* V *is bounded and surjective, and that the inequality* (2.3) *concerning to the adjoint operator* V^* *holds with* $0<A,B<\infty$, *or equivalently the operator norm of* V^* *is bounded above (resp. below) by* $\sqrt{B}$ *(resp.* A*). From the second equality in* (2.5), *it follows immediately that the upper (resp. lower) frame bounds of the frame generated by* V *is* B *(resp.* A*).*

For such a class of frame, it is not hard to give the corresponding dual frame:

Lemma 2.14. *Let* $\{\phi_\lambda\}_{\lambda\in\Lambda}$ *be a frame generated by* $\mathrm{V}:\mathcal{K}\to\mathcal{H}$. *Then the dual frame* $\{\tilde{\phi}_\lambda\}_{\lambda\in\Lambda}$ *of* $\{\phi_\lambda\}_{\lambda\in\Lambda}$ *is generated by the operator* $\tilde{\mathrm{V}}:\mathcal{K}\to\mathcal{H}$, *which has the form:*

$$\tilde{\mathrm{V}} = \tilde{\mathrm{V}}_{MP} + \mathrm{W}\mathrm{P}_{\ker(\mathrm{V})} = \tilde{\mathrm{V}}_{MP} + \mathrm{W}(Id_{\mathcal{H}} - \mathrm{V}^*\tilde{\mathrm{V}}_{MP}), \tag{2.6}$$

where $\tilde{\mathrm{V}}_{MP}:\mathcal{K}\to\mathcal{H}$ *denotes the Moore-Penrose pseudoinverse of* V^*, *i.e.* $\tilde{\mathrm{V}}_{MP} := (\mathrm{V}\mathrm{V}^*)^{-1}\mathrm{V}$, *and* $\mathrm{W}:\mathcal{K}\to\mathcal{H}$ *is any bounded operator.*

3. Fourier analysis on LCA groups

This chapter is devoted to the Fourier analysis on LCA groups. We first give some introduction to this class of groups. We shall introduce the notion of dual group and the notion of Haar measure, both constitute fundamentals for the Fourier transform of square-integrable functions on LCA groups. For self-containedness, the topological structure of the dual group of an LCA group shall be discussed intensively. In particular, this aims to show, that by an appropriate choice of the topology (compact-open topology) of the dual group, which is in some sense conformable with the Fourier transform, the dual group of an LCA group is also an LCA group.

For later purposes, we will also see that the dual group of a quotient group $\mathcal{G}/H$, where $\mathcal{G}$ is an LCA group, and $H \leqslant \mathcal{G}$ is a closed subgroup, and that the dual group of a closed subgroup of an LCA group can be identified (in topological group theoretic sense) with a more convenient group, called the annihilator. The important statement due to Pontryagin and van Kampen, which states, to say roughly, that the dual group of a dual group of an LCA group can be seen as the LCA group itself, shall also be established in this chapter.

If not otherwise stated, all the considered groups are, for ease of notations, written multiplicatively. Usually, the identity of a group $\mathcal{G}$ is emphasized by the subscript: $1_{\mathcal{G}}$. Recall that, since we consider mostly topological groups, homomorphism (resp. monomorphism, isomorphism, epimorphism, embedding) has to be understand as topological group homomorphism (resp. - monomorphism, - isomorphism, - epimorphism, - embedding). Otherwise the adjective "algebraic" (resp. "topological") shall be added. To avoid trivialities, all subsets (also compact - and open -) are, if not otherwise stated, considered as non-empty

The statements and its corresponding proof found in this chapter are adopted up to some modifications from [42], [51], [24], and [25].

3.1. Basic notions

We begin by introducing the definition of locally compact Abelian groups:

Definition 3.1 (LCA-groups). *A locally compact abelian (or shortly: LCA) group is a locally compact topological group, which is abelian/commutative as a group.*

Recall that Hausdorff property is already contained in the definition of topological groups, and hence also in the definition of LCA groups. It is straightforward to see that the following elementary spaces are LCA groups:

Examples 3.1.

- *The additive group* $(\mathbb{R}, +)$ *equipped with the natural topology.*
- *The additive group* $(\mathbb{Z}, +)$ *equipped with the discrete topology.*
- *The multiplicative group* $(\mathbb{T}, \cdot)$ *equipped with the relative topology induced by the natural topology on* $\mathbb{R}$*, and the additive group* $(\mathbb{R}/\mathbb{Z}, +)$ *equipped with the quotient topology induced by the natural topology on* $\mathbb{R}$*. It is known that both LCA groups are isomorphic (by the isomorphism* $\mathbb{R}/\mathbb{Z} \ni [x] \mapsto e^{2\pi i x} \in \mathbb{T}$*) to each other. So they both can be seen as identical.*

- *The finite group $\mathbb{Z}/k\mathbb{Z}$, where $k \in \mathbb{Z}$ equipped with the discrete topology.*

To an LCA group $\mathcal{G}$, there corresponds a unique set of functions, each maps $\mathcal{G}$ structure-preservative and continuously to the multiplicative group of the 1-dimensional torus:

Definition 3.2 (Characters). *Let $\mathcal{G}$ be an LCA group. A function γ mapping from $\mathcal{G}$ to the multiplicative LCA group $\mathbb{T}$ is said to be a character of $\mathcal{G}$ if γ is a (continuous) homomorphism.*

With suitable operations, the set of characters of $\mathcal{G}$ forms an Abelian group:

Definition 3.3 ((Pontryagin) Dual Group). *Let the set of all characters $\mathcal{G}^\wedge$ of $\mathcal{G}$ be equipped with the multiplication between $\gamma_1, \gamma_2 \in \mathcal{G}^\wedge$, given by the pointwise multiplication:*

$$(\gamma_1\gamma_2)(x) := \gamma_1(x)\gamma_2(x), \quad \forall x \in \mathcal{G},$$

the inverse of $\gamma \in \mathcal{G}^\wedge$, which is given by the complex conjugation: $\gamma^{-1} := \overline{\gamma}$, and the identity, which is the characteristic function on $\mathbb{T}$, i.e. $1_{\mathcal{G}^\wedge} := \chi_{\mathbb{T}}$. Then $\mathcal{G}^\wedge$ is called the (Pontryagin) dual group of $\mathcal{G}$.

It is immediate to see that the dual group of an LCA group is an Abelian group. Furthermore, it is obvious that from the definition of the characters and the corresponding operations between them, the following holds:

- $\gamma(1_{\mathcal{G}}) = 1_{\mathcal{G}^\wedge}(x) = 1_{\mathbb{T}}$, for all $\gamma \in \mathcal{G}^\wedge$, and $x \in \mathcal{G}$.
- $\gamma(x^{-1}) = \gamma^{-1}(x) = (\gamma(x))^{-1} = \overline{\gamma(x)}$, $\forall \gamma \in \Gamma$, $x \in \mathcal{G}$.

In the following, examples of dual groups of some LCA groups are given:

Examples 3.2.

- *Consider the additive group $\mathbb{R}$, equipped with its natural topology. It can easily be shown, that the dual group of $\mathbb{R}$ consists of the mappings: $\gamma_y(\cdot) := e^{2\pi i y(\cdot)}$, for $y \in \mathbb{R}$.*
- *The dual group $(\mathbb{R}/\mathbb{Z})^\wedge$ consists of the mappings: $\gamma_y(\cdot) := e^{2\pi i y(\cdot)}$, for $y \in \mathbb{Z}$.*
- *$\mathbb{Z}^\wedge$ consists of the mappings $\gamma_y(\cdot) := e^{2\pi i y(\cdot)}$, $y \in \mathbb{R}/\mathbb{Z}$.*
- *The dual group of the finite cyclic group $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ consists of the mappings $\gamma_y(\cdot) := e^{2\pi i \frac{y(\cdot)}{N}}$, $y \in \mathbb{Z}_N$.*

It is desirable to give the dual group a topology such that it becomes an LCA group. In the following section, we give the notion of the so called compact-open topology, which exactly meets this desired requirement.

3.2. Topological Structure of Dual Groups

Let $\mathcal{G}$ be an LCA group. The dual group $\mathcal{G}^\wedge$ is equipped with the topology induced by the sub-base consisting of the open (by definition) sets of the form:

$$\mathcal{W}(K, U) := \{\gamma \in \mathcal{G}^\wedge : \gamma(K) \subseteq U\}, \quad K \subseteq \mathcal{G} \text{ compact}, \; U \subseteq \mathbb{T} \text{ open}, \; K, U \neq \varnothing. \tag{3.1}$$

This topology on $\mathcal{G}^\wedge$ is called the *compact-open topology*. Clearly, this topology can also be applied to any subset of the set of mappings between $\mathcal{G}$ and $\mathbb{T}$, or even between any two topological spaces X and Y. In particular, we shall later also apply this topology to the set of homomorphism between $\mathcal{G}$ and $\mathbb{T}$, i.e. $\mathrm{Hom}(\mathcal{G}, \mathbb{T})$. Notice that, since finite subsets are compact, the compact-open topology is finer than the pointwise convergence topology, or equivalently the product topology, given in (A.2).

It is more convenient for some of our purposes, to describe the compact-open topology on $\mathcal{G}^\wedge$ alternatively as follows. Given a character $\gamma_0 \in \mathcal{G}^\wedge$, the neighborhood base of γ_0 consists of the open sets:

$$\mathcal{W}^{'}(\gamma_0, K, \epsilon) := \{\gamma \in \mathcal{G}^\wedge : |\gamma_0(x) - \gamma(x)| < \epsilon,\ \forall x \in K\}, \quad K \subseteq \mathcal{G} \text{ compact},\ K \neq \varnothing,\ \epsilon > 0. \tag{3.2}$$

Before we show the fact that the dual group $\mathcal{G}^\wedge$ of an LCA group $\mathcal{G}$ is also an LCA group, presumed that $\mathcal{G}^\wedge$ is equipped with the compact-open topology, we first show in the following lemma that by this setting, $\widehat{\mathcal{G}}$ becomes a (Hausdorff) topological Abelian group.

Lemma 3.3. *Let $\mathcal{G}$ be an LCA group. The dual group $\mathcal{G}^\wedge$ endowed by compact-open topology is a topological Abelian group.*

Proof. The fact that $\mathcal{G}^\wedge$ is an Abelian group, is easy to show, and was already mentioned. It is not hard to see that by this choice of topology, $\mathcal{G}^\wedge$ becomes a Hausdorff space. Indeed, let $\gamma, \gamma^{'} \in \mathcal{G}^\wedge$, where $\gamma \neq \gamma^{'}$. Now, take $x \in \mathcal{G}$ s.t. $\gamma(x) \neq \gamma^{'}(x)$. Since $\mathbb{T}$ is Hausdorff, we can choose an open neighborhood $U \subseteq \mathbb{T}$ of $\gamma(x)$, and $U^{'} \subseteq \mathbb{T}$ of $\gamma^{'}(x)$, s.t. $U \cap U^{'} = \varnothing$. Clearly, $\mathcal{W}(\{x\}, U)$ and $\mathcal{W}(\{x\}, U^{'})$ is contained in the compact-open topology of $\mathcal{G}^\wedge$, since $\{x\}$ is obviously compact. Finally, it is not hard to see that they are both disjoint, which shows that $\mathcal{G}$ is Hausdorff.

Now, we show that the multiplication $\mathcal{G}^\wedge \times \mathcal{G}^\wedge \to \mathcal{G}^\wedge$ is continuous. Let $\gamma, \gamma^{'} \in \mathcal{G}^\wedge$. Take a basic open set $\mathcal{W}(\gamma\gamma^{'}, K, \epsilon)$ around $\gamma\gamma^{'}$, where $K \subseteq \mathcal{G}$ a non-empty compact subset, and $\epsilon > 0$. We choose the open set $\mathcal{W}^{'}(\gamma, K, \epsilon/2) \times \mathcal{W}^{'}(\gamma^{'}, K, \epsilon/2)$ around $(\gamma, \gamma^{'})$ in $\mathcal{G}^\wedge \times \mathcal{G}^\wedge$. We shall show that the multiplication maps $\mathcal{W}^{'}(\gamma, K, \epsilon/2) \times \mathcal{W}^{'}(\gamma^{'}, K, \epsilon/2)$ into $\mathcal{W}^{'}(\gamma\gamma^{'}, K, \epsilon)$. So, let $\psi \in \mathcal{W}^{'}(\gamma, K, \epsilon/2)$, and $\psi^{'} \in \mathcal{W}^{'}(\gamma^{'}, K, \epsilon/2)$. By triangle inequality, we obtain:

$$\begin{aligned} |\psi(x)\psi^{'}(x) - \gamma(x)\gamma^{'}(x)| &= |(\psi(x) - \gamma(x))\psi^{'}(x) + (\gamma^{'}(x) - \psi^{'}(x))\gamma(x)| \\ &\leqslant |\psi(x) - \gamma(x)| + |\gamma^{'}(x) - \psi^{'}(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

for all $x \in \mathcal{G}$. So, the multiplication is continuous at $(\gamma, \gamma^{'})$. Finally, since γ and $\gamma^{'}$ is chosen arbitrarily, it follows the multiplication is continuous on $\mathcal{G}^\wedge \times \mathcal{G}^\wedge$.

It remains now to show that the inversion is continuous. Let $\gamma \in \mathcal{G}^\wedge$, and take a non-empty compact $K \subseteq \mathcal{G}$, and $\epsilon > 0$. Then $\mathcal{W}^{'}(\gamma, K, \epsilon)$ is an open neighborhood of γ, contained in the neighborhood base of γ. For $\psi \in \mathcal{W}^{'}(\gamma, K, \epsilon)$, we compute:

$$|\psi^{-1}(x) - \gamma^{-1}(x)| = |\frac{\psi(x) - \gamma(x)}{\psi(x)\gamma(x)}| = |\psi(x) - \gamma(x)| < \epsilon,$$

for all $x \in \mathcal{G}$, which shows that $\mathcal{W}^{'}(\gamma, K, \epsilon)$ is mapped by the inversion to $\mathcal{W}^{'}(\gamma^{-1}, K, \epsilon)$, which shows that the inversion is continuous at γ. Noticing that γ is chosen arbitrarily, the desired statement is established. ∎

Since we now know that $\mathcal{G}^\wedge$ equipped with compact-open topology is a topological group, it is obviously enough to turn our attention to the neighborhood base of the identity in $\mathcal{G}^\wedge$, since all topological properties and continuity of a map (provided that the target of the map is also a topological group) can be shown only by considering this collection of subsets (prop. C.8, prop. C.7). In particular we shall consider the following neighborhood base of the identity formed by the open sets of the form:

$$\mathcal{W}(K, U) := \{\gamma \in \mathcal{G}^\wedge : \gamma(K) \subseteq U\},$$

where $K \subseteq \mathcal{G}$ is a non-empty compact subset, and $U \subseteq \mathbb{T}$ a neighborhood of the identity 1 in $\mathbb{T}$.

In the following, we will see, by observing the topological structure of the neighborhood of the identity in the dual group, that this group is indeed an LCA group. But first, we need an auxiliary statement concerning to the structure of the set of homomorphisms between $\mathcal{G}$ and $\mathbb{T}$. Recall that $\mathbb{T}^{\mathcal{G}}$ denotes the set of mappings between $\mathcal{G}$ and $\mathbb{T}$ equipped with the product topology, or equivalently, the pointwise convergence topology.

Lemma 3.4. *Let $\mathcal{G}$ be an LCA group. The set of homomorphic functions between $\mathcal{G}$ and $\mathbb{T}$ is compact in $\mathbb{T}^{\mathcal{G}}$ equipped with product topology.*

Proof. We already know that $\mathbb{T}^{\mathcal{G}}$ equipped with the product topology is by Tychonov's theorem compact. Hence, to show the compactness of $\mathrm{Hom}(\mathcal{G},\mathbb{T})$ in $\mathbb{T}^{\mathcal{G}}$, it suffices to establish the closeness of $\mathrm{Hom}(\mathcal{G},\mathbb{T})$ in $\mathbb{T}^{\mathcal{G}}$. So by some computations:

$$\begin{aligned}\mathrm{Hom}(\mathcal{G},\mathbb{T}) &= \bigcap_{h,g\in\mathcal{G}}\{\gamma\in\mathbb{T}^{\mathcal{G}} : \gamma(h+g)=\gamma(h)f(g)\} = \bigcap_{x,y\in\mathcal{G}}\{\gamma\in\mathbb{T}^{\mathcal{G}} : \mathrm{pr}_{h+g}(\gamma)=\mathrm{pr}_h(\gamma)\mathrm{pr}_g(\gamma)\}\\ &= \bigcap_{h,g\in\mathcal{G}}\{\gamma\in\mathbb{T}^{\mathcal{G}} : (\mathrm{pr}_{h+g}^{-1}\circ(\mathrm{pr}_h\mathrm{pr}_g))(\gamma)=1_{\mathcal{G}}\},\end{aligned}$$

where $1_{\mathcal{G}}$ denotes the constant function, mapping whole $\mathcal{G}$ to the identity of $\mathbb{T}$, and $\mathrm{pr}_h\mathrm{pr}_g$ denotes the mapping $(\mathrm{pr}_h\mathrm{pr}_g)(f) = \mathrm{pr}_h(f)\mathrm{pr}_g(f)$. Notice that $\mathrm{pr}_h\mathrm{pr}_g$ can be seen as the composition of the mapping $\mathbb{T}^{\mathcal{G}} \ni f \mapsto (\mathrm{pr}_h(f),\mathrm{pr}_g(f)) \in \mathbb{T}$ and the multiplication in $\mathbb{T}$. Since all projections from product of topological spaces equipped with product topology are continuous, multiplication in $\mathbb{T}$ is continuous, and composition of continuous functions yields a continuous function, it follows that $\mathrm{pr}_h\mathrm{pr}_g : \mathbb{T}^{\mathcal{G}} \to \mathbb{T}$ is continuous. Notice that the subset A of $\mathbb{T}$, for which $\mathrm{pr}_{h+g}^{-1}(A) = 1^{\mathcal{G}}$ holds, is the singleton $\{1\}$, which is clearly closed. The fact that the preimage of the closed set $\mathrm{pr}_{h+g}^{-1}(A)$ under the continuous mapping $\mathrm{pr}_h\mathrm{pr}_g$ is closed shows that the set $\{\gamma\in\mathbb{T}^{\mathcal{G}} : (\mathrm{pr}_{h+g}^{-1}\circ(\mathrm{pr}_h\mathrm{pr}_g))(\gamma)=1_{\mathcal{G}}\}$, for all $x,y\in\mathcal{G}$, is closed. So basically, $\mathrm{Hom}(\mathcal{G},\mathbb{T})$ is arbitrary intersection of closed sets, and hence closed w.r.t. the topology on $\mathbb{T}^{\mathcal{G}}$. Since $\mathbb{T}^{\mathcal{G}}$ is compact, it follows immediately that $\mathrm{Hom}(\mathcal{G},\mathbb{T})$ is compact in $\mathbb{T}^{\mathcal{G}}$. ∎

By means of the topological structure of $\mathrm{Hom}(\mathcal{G},\mathbb{T})$ in $\mathbb{T}^{\mathcal{G}}$, the local compactness of the dual group of an LCA group can be established. In the following theorem, we shall give a sketch of the proof of this property. For ease of notations, we denote $\mathrm{Hom}(\mathcal{G},\mathbb{T})$ by $\mathcal{G}^*$.

Theorem 3.5. *Let $\mathcal{G}$ be an LCA group. Then $\mathcal{G}^\wedge$, equipped with compact-open topology, is also an LCA group*

Sketch of Proof. Define for $k\in\mathbb{N}$, $\Lambda_k := p((-\frac{1}{3k},\frac{1}{3k}))$, where $p:\mathbb{R}\to\mathbb{T}$ is the composition of the canonical quotient mapping between $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$, and the canonical topological group isomorphism between $\mathbb{R}/\mathbb{Z}$ and $\mathbb{T}$. It can be shown that $\{\Lambda_k\}_{k\in\mathbb{N}}$ is a neighborhood base of $0_{\mathbb{R}}$, w.r.t. the natural topology. Notice that by definition of quotient topology, and from the fact that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to $\mathbb{T}$ by the corresponding homeomorphism, it follows that $\{p(\Lambda_k)\}_{k\in\mathbb{N}}$ is a neighborhood base of the identity $1_{\mathbb{T}}$ of $\mathbb{T}$. The following technical statement can be established:

$$\forall k\in\mathbb{N}: \quad x\in\Lambda_k \Leftrightarrow x,x^2,\ldots,x^k\in\Lambda_1. \tag{3.3}$$

From above statement, one can imply the fact that a homomorphism between $\mathcal{G}$ and $\mathbb{T}$ is continuous if and only if the preimage of Λ_1 under this homomorphism is a neighborhood of $1_{\mathcal{G}}$, or equivalently:

$$\mathcal{W}_{\mathcal{G}^*}(U,\Lambda_1) = \mathcal{W}_{\mathcal{G}^\wedge}(U,\Lambda_1), \quad \forall U \text{ neighborhood of } 1_{\mathcal{G}}. \tag{3.4}$$

To proof this statement, let $\gamma\in\mathcal{G}^*$. Suppose that $\gamma^{-1}(\Lambda_1)$ is a neighborhood of $1_{\mathcal{G}}$. From the definition of neighborhoods, it follows that there exists an open set $U\subseteq\Lambda_1$ containing $1_{\mathcal{G}}$. From elementary property of topological groups, it follows that there exists a neighborhood V of $1_{\mathcal{G}}$ s.t. $V^k \subseteq U$. Fix such a $k \in \mathbb{N}$. From the homomorphic property of γ, it holds that for $l \in [k]$, $(\gamma(x))^l \in \Lambda_1$, $\forall x \in V$. By (3.3), it follows that $\gamma(x) \in \Lambda_k$, $\forall x \in V$, and hence $\gamma(V) \subseteq \Lambda_k$, for each $k \in \mathbb{N}$. Since $\{\Lambda_k\}_{k\in\mathbb{N}}$ forms a neighborhood base of $1_{\mathbb{T}}$, it holds that γ is continuous at $1_{\mathcal{G}}$, and hence on whole $\mathcal{G}$. Finally, the statement (3.4) is shown. So now we have seen, that it is unnecessary to make a difference between the sets $\mathcal{W}_{\mathcal{G}*}(U, V')$ and $\mathcal{W}_{\mathcal{G}^\wedge}(U, V')$, for each neighborhood U of $1_{\mathcal{G}}$, and each subset V' of Λ_1. This sets shall simply be denoted by $\mathcal{W}(U, V')$.

Now, we show that the following statement holds:

$$\forall K \subseteq \mathcal{G} \text{ compact, } C \subseteq \mathbb{T} \text{ closed}: \quad \mathcal{W}_{\mathcal{G}*}(K, C) \text{ compact,} \tag{3.5}$$

■

where the compactness and closeness is w.r.t. the topology of $\mathcal{G}^*$ induced from $\mathbb{T}^{\mathcal{G}}$. It is not hard to show that $\mathcal{W}_{\mathcal{G}*}(K, C) = \bigcap_{x\in K}(\mathrm{pr}_x^{-1}(C) \cap \mathcal{G}^*)$, where pr_x denotes the canonical projection from topological product space. Since $(\mathrm{pr}_x^{-1}(C) \cap \mathcal{G}^*)$ is closed in $\mathcal{G}^*$, (notice that preimage of closed set under continuous function is closed and from lemma 3.4 $\mathcal{G}^*$ is closed), it follows immediately that $\mathcal{W}_{\mathcal{G}*}(K, C)$ is closed in $\mathcal{G}^*$, and hence compact, since $\mathcal{G}^*$ is compact by lemma 3.4. So the statement is established.

Now we come to our actual purpose to sketch the proof of the local compactness of the dual group. Since we have seen that $\mathcal{G}^\wedge$ is a (Hausdorff) topological group, it remains to show that there exists a compact neighborhood of $1_{\mathcal{G}^\wedge}$. We claim that $\mathcal{W}_1 := \mathcal{W}_{\mathcal{G}^\wedge}(\overline{U}, \overline{\Lambda_4})$ is compact w.r.t. the compact open topology, for each U neighborhood of $1_{\mathcal{G}}$. Notice that since $\Lambda_4 \subseteq \Lambda_1$, $\mathcal{W}_{\mathcal{G}^\wedge}(\overline{U}, \overline{\Lambda_4}) = \mathcal{W}_{\mathcal{G}*}(\overline{U}, \overline{\Lambda_4})$ by (3.3). Hence, we can apply (3.5), and imply that $\mathcal{W}_1$ is compact w.r.t. to the topology of $\mathcal{G}^*$. It remains now to show that the subspace topology (call $\tau^\wedge$) on $\mathcal{W}_1$ inherited from $\mathcal{G}^\wedge$ coincides with the subspace topology (call τ^*) on $\mathcal{W}_1$ inherited from $\mathcal{G}^*$. Since $\tau^\wedge$ is finer than τ^* (follows from the fact that compact-open topology is finer than pointwise convergence topology), it is enough to show that $\tau^* \subseteq \tau^\wedge$, which is fairly easy to show.

In case that some informations about the topological structure of the considered LCA group is available, one can make a guess about the topological structure of its dual group, as shown in the following:

Theorem 3.6. *Let $\mathcal{G}$ be a LCA group. Then the following holds:*

(a) If $\mathcal{G}$ is compact, then $\mathcal{G}^\wedge$ is discrete.

(b) If $\mathcal{G}$ is discrete, then $\mathcal{G}^\wedge$ is compact.

Proof. Let $\mathcal{G}$ be compact. Consider the neighborhood $\mathcal{W}(\mathcal{G}, U)$, with U is an open set sufficiently small in $\mathbb{T}$ s.t. no non-trivial subgroup of $\mathbb{T}$ contained therein, of the identity $1_{\mathcal{G}^\wedge}$. For any $\gamma \in \mathcal{W}(\mathcal{G}, U)$, since γ is a homomorphism, $\gamma(\mathcal{G})$ is a subgroup of $\mathbb{T}$, but only the trivial subgroup is contained in U. Hence it must hold: $\gamma \equiv 1_{\mathcal{G}}$, $\mathcal{W}(\mathcal{G}, U) = \{1_{\mathcal{G}^\wedge}\}$, and correspondingly $1_{\mathcal{G}^\wedge}$ is discrete, which shows the first statement.

For the proof of the second statement: let $\mathcal{G}$ be discrete. It is clear that every function mapping from discrete space to any topological space is continuous. So $\mathcal{G}^\wedge$ coincides with the set $\mathrm{Hom}(\mathcal{G}, \mathbb{T})$, which is a subset of $\mathbb{T}^{\mathcal{G}}$. In lemma 3.4, we have already seen that $\mathrm{Hom}(\mathcal{G}, \mathbb{T})$, and hence $\mathcal{G}^\wedge$ is compact w.r.t. to the topology of $\mathbb{T}^{\mathcal{G}}$. It now remains to show that the topology on $\mathrm{Hom}(\mathcal{G}, \mathbb{T})$ induced from $\mathbb{T}^{\mathcal{G}}$ coincides with the compact-open topology. Recall that the subsets $\bigcap_{x\in F} \mathrm{pr}_x^{-1}(U)$, where $F \subseteq \mathcal{G}$ finite, and $U \subseteq \mathbb{T}$ open, constitute a base for the product topology on $\mathbb{T}^{\mathcal{G}}$. By easy reformulations:

$$\begin{aligned}
(\bigcap_{x\in F} \mathrm{pr}_x^{-1}(U)) \cap \mathrm{Hom}(\mathcal{G}, \mathbb{T}) &= \{\gamma \in \mathrm{Hom}(\mathcal{G}, \mathbb{T}) : \mathrm{pr}_x(\gamma) \in U, \ \forall x \in F\} \\
&= \{\gamma \in \mathrm{Hom}(\mathcal{G}, \mathbb{T}) : \gamma(x) \in U, \ \forall x \in F\} = \mathcal{W}(F, U),
\end{aligned}$$

which shows that the base of the topology on $\mathrm{Hom}(\mathcal{G}, \mathbb{T})$ induced from $\mathbb{T}^{\mathcal{G}}$ conincides with the base of the compact open topology on $\mathrm{Hom}(\mathcal{G}, \mathbb{T})$. Hence $\mathrm{Hom}(\mathcal{G}, \mathbb{T}) = \mathcal{G}^\wedge$ is compact w.r.t. the compact open topology. ∎

Given a finite product of LCA groups, one can easily compute the dual of this LCA group, provided that the dual of each LCA group in this finite collection is known, as done in the following theorem:

Theorem 3.7. *Let $\{G_i\}_{i\in[n]}$ be a finite collection of LCA groups. Then the dual group of the product of $\{G_i\}_{i\in[n]}$ can be identified with the product of the dual groups $\{\widehat{G}_i\}_{i\in[n]}$, by the identification:*

$$(\mathcal{G}_1 \times \cdots \times \mathcal{G}_n)^\wedge \ni \gamma_1 \cdots \gamma_n \longleftrightarrow (\gamma_1, \ldots, \gamma_n) \in \mathcal{G}_1^\wedge \times \cdots \times \mathcal{G}_n^\wedge$$

Proof. It is sufficient to give the proof for the case $n = 2$. The desired statement can then be easily shown by induction.

Now, let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$. Define the mapping $\phi : \mathcal{G}_1^\wedge \times \mathcal{G}_2^\wedge \to (\mathcal{G}_1 \times \mathcal{G}_2)^\wedge$, by $\phi(\gamma_1, \gamma_2)(x_1, x_2) := \gamma_1(x_1)\gamma_2(x_2)$, $(x_1, x_2) \in \mathcal{G}_1 \times \mathcal{G}_2$. By computations, It is not hard to see that ϕ is an algebraic homomorphism. By elementary reformulations of $\ker \phi$, one immediately observes that $\ker \phi$ contains only the cartesian product of the identity in $\mathcal{G}_1$ and $\mathcal{G}_2$, and hence ϕ is injective.

Surjectivity of ϕ is easy to show: Take $\gamma \in (\mathcal{G}_1 \times \mathcal{G}_2)^\wedge$. One can of course decompose γ for all $x_1 \in \mathcal{G}_1$ and $x_2 \in \mathcal{G}_2$ into $\gamma(x_1, x_2) = \gamma(x_1, 1_{\mathcal{G}_2})\gamma(1_{\mathcal{G}_1}, x_2)$, where $1_{\mathcal{G}_1} \in \mathcal{G}_1$ and $1_{\mathcal{G}_2} \in \mathcal{G}_2$ are identities. Now define $\gamma_1(x_1) := \gamma(x_1, 1_{\mathcal{G}_1})$, $\forall x_1 \in \mathcal{G}_1$, and also $\gamma_2(x_2) := \gamma(1_{\mathcal{G}_1}, x_2)$, $\forall x_2 \in \mathcal{G}_2$. It is elementary to see that γ_1 and γ_2 are indeed characters on $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

Now, we show that ϕ is continuous. Let $\mathcal{W}(K, U)$ be a basic open neighborhood of the identity in $H =: (\mathcal{G}_1 \times \mathcal{G}_2)^\wedge$, where $K \subseteq \mathcal{G}_1 \times \mathcal{G}_2$ is compact, and $U \subseteq \mathbb{T}$ open neighborhood of the identity in $\mathbb{T}$. The canonical projections $\phi_1 : \mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{G}_1$ and $\phi_2 : \mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{G}_2$ are clearly continuous. The sets $\phi_1(K)$ and $\phi_2(K)$ are hence compact in $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, since they are each an image of a compact set under continuous mapping. Choose a neighborhood V of the identity 1 in $\mathbb{T}$ s.t. $V^2 \subseteq U$, and notice that $\mathcal{W}_{\mathcal{G}_1^\wedge}(K, V) \times \mathcal{W}_{\mathcal{G}_2^\wedge}(K, V))$ is a (open) neighborhood of the identity in $\mathcal{G}_1^\wedge \times \mathcal{G}_2^\wedge$. By detailed observations, one can see that by our choice of V, $\phi(\mathcal{W}_{\mathcal{G}_1^\wedge}(K, V) \times \mathcal{W}_{\mathcal{G}_2^\wedge}(K, V)) \subseteq \mathcal{W}(K, U)$, and accordingly ϕ is continuous.

Finally, we show now that ϕ is open. Let $\mathcal{W}_{\mathcal{G}_1^\wedge}(K_1, U_1)$ be an open neighborhood of the identity $1_{\mathcal{G}_1^\wedge} \in \mathcal{G}_1^\wedge$, and $\mathcal{W}_{\mathcal{G}_2^\wedge}(K_2, U_2)$ be an open neighborhood of the identity $1_{\mathcal{G}_2^\wedge} \in \mathcal{G}_2^\wedge$, where $K_1 \subseteq \mathcal{G}_1$, $K_2 \subseteq \mathcal{G}_2$ are compact, and $U_1, U_2 \subseteq \mathbb{T}$ are open neighborhoods of the identity $1 \in \mathbb{T}$. Observe that $K := (K_1 \cup \{1_{\mathcal{G}_1}\}) \times (K_2 \cup \{1_{\mathcal{G}_2}\})$ is compact in $\mathcal{G}_1)^\times \mathcal{G}_2$, and $U := U_1 \cap U_2$ is an open neighborhood of 1 in $\mathbb{T}$. We now claim that: $\mathcal{W}(K, U) \subseteq \phi(\mathcal{W}_{\mathcal{G}_1^\wedge}(K_1, U_1) \times \mathcal{W}_{\mathcal{G}_2^\wedge}(K_2, U_2))$. Let $\gamma \in \mathcal{W}(K, U)$. We can write γ as the image of the characters $\gamma_1 := \gamma((\cdot), 1_{\mathcal{G}_2})$ and $\gamma_2 := \gamma(1_{\mathcal{G}_1}, (\cdot))$ on $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, under the mapping ϕ, which shows that the claim holds true. Hence, ϕ is open. ∎

This concept is illustrated by the following examples concerning to elementary LCA groups:

Examples 3.8. *Consider the finite product of the euclidean space $\mathbb{R}^N$, where $N \in \mathbb{N}$. We already know that the dual group of $\mathbb{R}$ consists the mappings $\gamma_y(\cdot) = e^{2\pi i y(\cdot)}$, for $y \in \mathbb{R}$, and can be identified with $\mathbb{R}$. So, the group $(\mathbb{R}^N)^\wedge$ consists of the elements $(\gamma_{y_1}, \ldots, \gamma_{y_N})$, where $\gamma_{y_k} := e^{2\pi i y_k(\cdot)}$, and $y_k \in \mathbb{R}$, $\forall k \in [N]$. By thm. 3.7, the dual group $(\mathbb{R}^N)^\wedge$ of $\mathbb{R}^N$ consists of the mappings $\gamma_\omega = \prod_{k\in[N]} \gamma_{\omega_k}$, where $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$. By this reason, $(\mathbb{R}^N)^\wedge$ can be identified with $\mathbb{R}^N$. One can proceed in the same way to find out the dual group of $(\mathbb{R}/\mathbb{Z})^N$, the dual group of $\mathbb{Z}^N$, the dual group of $(\mathbb{Z}_n)^N$, and the dual group of the product of finite mixtures of $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}$, and $\mathbb{Z}/k\mathbb{Z}$, respectively.*

3.3. Separating Property of Dual Groups

A fundamental property of the dual group $\mathcal{G}^\wedge$ of an LCA group $\mathcal{G}$ is that it constitute a so-called separating set of $\mathcal{G}$. For self-containedness, we give the notion of separating sets in the following definition:

Definition 3.4 (Separating set). *Let X, Y be two sets. The set of functions $S \subseteq Map(X, Y)$ is said to be separating set of X or to separate the points of X if:*

$$\forall x, x^{'} \in X,\ x \neq x^{'} : \exists f \in S : \quad f(x) \neq f(x^{'}).$$

Since a character of an LCA group $\mathcal{G}$ is a homomorphism between topological groups, to proof that $\mathcal{G}^\wedge$ is a separating set of $\mathcal{G}$, it obviously suffices to show that for $x \in \mathcal{G} \backslash \{1_\mathcal{G}\}$, there exists a $\gamma \in \mathcal{G}^\wedge$, s.t. $\gamma(x) \neq 1_\mathbb{T}$. In particular, the separating property of the dual group of an non-trivial LCA group ensures that the dual group of an LCA group is also non-trivial.

In case that $\mathcal{G}$ is compact, the desired property of its dual group is relatively easy to see:

Lemma 3.9 (Peter-Weyl's Theorem). *Let $\mathcal{G}$ be an LCA group. If $\mathcal{G}$ is compact, then $\mathcal{G}^\wedge$ separates the points of $\mathcal{G}$*

Consequently, one can show the following useful characterization:

Lemma 3.10. *Let $\mathcal{G}$ be a compact LCA group, and H be a subgroup of the dual group $\mathcal{G}^\wedge$. If H separates points of $\mathcal{G}$, then $H = \mathcal{G}^\wedge$.*

Proof. For proof, see e.g. Corollary 1 in [42] ∎

The following statement says roughly that the the characters of a compact subgroup of an LCA group $\mathcal{G}$ are inherited from the dual group of $\mathcal{G}$:

Corollary 3.11. *Let $\mathcal{G}$ be an LCA group, and $K \subseteq \mathcal{G}$ compact. Then for every $\gamma \in K^\wedge$, there exists an $\tilde{\gamma} \in \mathcal{G}^\wedge$ s.t. $\tilde{\gamma}|_K = \gamma$*

Proof. Define the subset A of $K^\wedge$, by $A := \{\gamma \in K^\wedge : \exists \tilde{\gamma} \in \mathcal{G}^\wedge \text{ s.t. } \tilde{\gamma}|_K = \gamma\}$. Since $\mathcal{G}^\wedge$ separates points of $\mathcal{G}$ (thm. 3.14), it follows immediately that A separates points of K. Finally, lemma 3.10 asserts that $A = K^\wedge$. ∎

In case that $\mathcal{G}$ is discrete, one obtains the following slightly weaker form of lemma 3.10

Lemma 3.12. *Let $\mathcal{G}$ be discrete LCA group. If H is a subgroup of $\mathcal{G}^\wedge$ which separates points of $\mathcal{G}$, then H is dense in $\mathcal{G}^\wedge$.*

Proof. See e.g. Prop. 32 in [42]. ∎

Before we proof the separating property of the dual group of a general LCA groups, we need first the following lemma:

Lemma 3.13. *Let H be a subgroup of an Abelian group $\mathcal{G}$, and D is an divisible Abelian group. Given a mapping $\phi : H \to D$. If ϕ is a homomorphism, then ϕ can be extended to a homomorphism between $\mathcal{G}$ and D, i.e.: there exists a homomorphism $\tilde{\phi} : \mathcal{G} \to D$ s.t. $\tilde{\phi}\Big|_D = \phi$.*

Furthermore, if in addition the following holds:

- *$\mathcal{G}$ and D are topological groups,*
- *H is a subgroup of $\mathcal{G}$ carrying the subspace topology,*
- *ϕ is continuous,*

then $\tilde{\phi}$ is a continuous homomorphism between $\mathcal{G}$ and D.

Proof. We proof the first statement by "induction". Let $x \notin H$, define the subgroup H_0 of H by $H_0 := \{x^n h : h \in H,\ n \in \mathbb{Z}\}$. We differ two cases:

(i) If $\forall n \in \mathbb{N}$, $x^n \notin H$ (suspiciously, x^{-n}, $n \in \mathbb{N}$, are also not cantained in H), then define a function $\phi_0 : H_0 \to D$, which "annihilate" the $\{x^n : n \in \mathbb{Z}\}$ by $\phi_0(x^n h) = \phi(h)$, $\forall n \in \mathbb{Z}\backslash\{0\}$. One can easily check that ϕ is well-defined, a homomorphism, and $\phi_0|_H = \phi$.

(ii) Assume[1] that for some $n \in \mathbb{N}\backslash\{1\}$, $x^n \in H$. Take the least $k \in \mathbb{N}$, $k \geqslant 2$, s.t. $x^k \in H$. Clearly $\phi(x^k) = d$, where $d \in D$. Since D is divisible, there exists $a \in D$ s.t. $a^k = d$. So now we can define $\phi_0 : H_0 \to D$, with $\phi_0(x^n h) = \phi(h)a^n$, $\forall n \in \mathbb{Z}$. So one immediately see that ϕ_0 fulfilled the desired statement.

One may continue iteratively for $l = 1, 2, \dots$ to define $\phi_l : H_l \to D$, where $H_l = \langle H_{l-1} \cup x\rangle$, where $x \notin H_{l-1}$, as analogue to above approach to define ϕ_l. One can easily check that the set of tuple $\{(H_l, \phi_l)\}_{l \in \mathbb{N}}$ forms a chain, with order: $(H_i, \phi_i) \leqslant (H_j, \phi_j) :\Leftrightarrow H_i \leqslant H_j$ and $\phi_j|_{H_i} = \phi_i$. By Zorn's lemma, a "maximal" tuple exists. In particular this tuple must be (H_∞, ϕ_∞), where $H_\infty = \mathcal{G}$, and correspondingly $\phi_\infty = \tilde{\phi}$.

For the second statement: Let be in addition $\mathcal{G}$ and D topological groups, H a subgroup of $\mathcal{G}$ carrying the subspace topology (notice that open subgroups of an LCA group is automatically closed), and ϕ continuous. Consider the extension $\tilde{\phi}$ of ϕ to $\mathcal{G}$. Obviously, $\tilde{\phi}$ must be continuous, since its restriction ϕ to the closed subset $\mathcal{H}$ is continuous. ∎

Now, we are able to show the separation property of the dual group of a general LCA group:

Theorem 3.14. *Let $\mathcal{G}$ be an LCA group. Then $\mathcal{G}^\wedge$ separates the points of $\mathcal{G}$.*

Proof. Take $x \in \mathcal{G}$, not equal the identity, and a compact neighborhood V of the identity. From (C.4), it follows that $\mathcal{H} := \langle V \cup \{x, x^{-1}\}\rangle$ is an open compactly generated subgroup of $\mathcal{G}$. From theorem 3.16, it follows that there exists $K \leqslant H$, with K topologically isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}_0$, s.t. H/K is compact s.t. $(V \cup \{x, x^{-1}\}) \cap K = \neq$. Let q be the canonical quotient map between H and H/K. Clearly by this construction, $\mathrm{q}(x) \neq 1_{H/K}$.

Applying lemma 3.9 to the compact abelian group H/K, it follows that there exists a continuous homomorphism $\phi : H/A \to \mathbb{T}$ s.t. $\phi(\mathrm{q}(x)) \neq 1_{\mathbb{T}}$. Now, notice that $\phi \circ \mathrm{q}$, as a composition of continuous homomorphism, is a continuous homomorphism from H to $\mathbb{T}$. By noticing that $\mathbb{T}$ is divisible, and by applying lemma 3.13, it follows that $\phi \circ \mathrm{q}$ can be extended to a continuous homomorphism $\gamma : \mathcal{G} \to \mathbb{T}$. Since $\phi \circ \mathrm{q}(x) \neq 1_{\mathbb{T}}$, it clearly holds also $\gamma(x) \neq 1_{\mathbb{T}}$, which gives the desired statement. ∎

In the next section, we concern ourselves with structure of LCA groups. We shall see that almost all LCA groups can be "seen" as a product of "familiar" groups, which may make the analysis on this class of groups easier.

3.4. Structure Theorems of LCA groups

First, we start with the following fundamental, but not trivial, statement due to Weil. A group ßG is said to be cyclic group if it is generated by single element, i.e. $\mathcal{G} = \langle x\rangle$, for some $x \in \mathcal{G}$.

Lemma 3.15 (Weil's Lemma). *Let $\mathcal{G}$ be an LCA group. If $\mathcal{G}$ has a dense cyclic group, then $\mathcal{G}$ is either compact or discrete.*

[1] e.g. in the case $\mathcal{G} = \mathbb{Z}$ additive , $H = 2\mathbb{Z}$, $x = 3$, it holds $2x = 6 \in 2\mathbb{Z}$

Proof. See e.g. Thm. 2.3.2 in [51] ■

By means of Weil's Lemma, one can ensure the existence of a subgroup of an LCA group, provided that the LCA group is compactly generated, called uniform lattice, which we shall intensively consider later. Recall that an LCA group is said to be compactly generated, if it is algebraically generated by a compact subset.

Theorem 3.16. *Let $\mathcal{G}$ be a compactly generated LCA group. Then $\mathcal{G}$ has a subgroup Λ which is topological group isomorphic to $\mathbb{Z}^n$, for some $n \in \mathbb{N}_0$ s.t. $\mathcal{G}/\Lambda$ is compact.*

Proof. Let $\mathcal{G}$ be an LCA group written multiplicatively, and let $\mathcal{G}$ be representable as $\mathcal{G} = \bigcup_{i\in\mathbb{N}} V^i$, where V is a compact neighborhood of the identity 0 of $\mathcal{G}$. As a multiplication of 2 compact sets, V^2 is compact. Hence, it follows that there exist finitely many $g_1, \ldots, g_m \in \mathcal{G}$ s.t. the finite collection of subsets $\{(g_iV)\}_{i\in[m]}$ covers V^2, i.e. $V^2 \subseteq \bigcup_{i=1}^m (g_iV)$.

Let $\Lambda := \langle g_1, \ldots, g_m\rangle$ be the subgroup generated by $\{g_i\}_{i\in[m]}$. We claim that:

$$\mathcal{G} = V\Lambda. \tag{3.6}$$

To show this claim, notice that $V^i \subseteq V\Lambda$ for $i \in \{1, 2\}$, since Λ contains the identity and hence $V^1 = V \subseteq$, and since, as it has been shown, $V^2 \subseteq \bigcup_{i=1}^m (g_iV) \subseteq V + \Lambda$. By induction and previous observations, one can immediately see that $V_n \subseteq VH$, for all $n \in \mathbb{N}$, and therefore this claim holds.

Write $\Lambda = \prod_{i\in[m]} \Lambda_i$, where $\Lambda_i := \langle g_i\rangle$, $i \in [m]$. From lemma 3.15, it follows that for all $i \in [n]$, either $\Lambda_i \cong \mathbb{Z}$ or $\overline{\Lambda_i}$ is compact.

Suppose that $\overline{\Lambda_i}$ is compact $\forall i \in [n]$. Then, as a multiplication of finite number of compact sets $\{\overline{\Lambda_i}\}$, $\overline{\Lambda}$ is compact. Notice that it holds $\mathcal{G} = V\overline{\Lambda}$, and $\mathcal{G}$ is compact, since V and $\overline{\Lambda}$ are compact. So the proposition is true with $n = 0$.

Suppose now that there exists some $\mathcal{I} \subseteq [m]$ s.t. $\Lambda_i \cong \mathbb{Z}$, $\forall i \in \mathcal{I}$. Write $\mathcal{G} = \tilde{V}\Lambda'$, where $\tilde{V} := V\prod_{i\in[n]\setminus\mathcal{I}} \overline{\Lambda_i}$, and $\Lambda' := \prod_{i\in\mathcal{I}} \overline{\Lambda_i}$. Obviously, $\tilde{V}$ is compact, and $\Lambda' \cong \mathbb{Z}^l$, for an $l \in \mathbb{N}$, which shows the desired statement. ■

Furthermore, above statement can be sharpened in the way, that if a compact subset V algebraically generated $\mathcal{G}$, that the uniform lattice Λ can be chosen s.t. $V \cap \Lambda = \varnothing$ (see e.g. Lemma 2.4.2 in [51]).

The following Thm. states, in some sense, that each compactly generated group can be reduced to an "elementary" LCA group. Formally, an elementary LCA group $\mathcal{G}$ is defined as an LCA group, which isomorphic to $\mathbb{R}^N \times \mathbb{Z}^M \times \mathbb{T}^{N'} \times F$, for some F is a finite discrete abelian group, and $N, M, N' \in \mathbb{N}_0$.

Theorem 3.17. *Let $\mathcal{G}$ be an LCA group. If $\mathcal{G}$ is compactly generated, then it has a compact subgroup K s.t. $\mathcal{G}/K$ is isomorphic to an elementary group.*

Proof. See e.g. Proposition 34. [42] ■

Of ancilliary interests to us, in case that there is no additional information on the considered group, one can roughly guess its structure:

Theorem 3.18 (First Structure Theorem). *Let $\mathcal{G}$ be an LCA group. Then there exist a $n \in \mathbb{N}_0$ and an LCA group H s.t. $\mathcal{G}$ is isomorphic to $\mathbb{R}^N \times H$, where $\mathcal{H}$ contains an open compact subgroup K.*

Proof. See e.g. Thm. 2.4.1 in [51] ■

We shall later by some reasons, restrict ourselves to LCA groups, which are algebraically generated by compact subsets. In this case, one can give a relatively good hint on its structure:

Theorem 3.19 (Second Structure Theorem). *Let $\mathcal{G}$ be a compactly generated LCA group. Then there exist $n, m \in \mathbb{N}_0$ and a compact group K s.t. $\mathcal{G}$ is topological group isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m \times K$.*

We say two topological groups $\mathcal{G}$ and $\mathcal{G}^{'}$ is said to be locally identifiable, or more formally: locally isomorphic, if there exists an open neighborhood $U \subseteq \mathcal{G}$ and $U^{'} \subseteq \mathcal{G}^{'}$ of the identity, and a homeomorphism $\phi : U \to U^{'}$ s.t. $\phi(xy) = \phi(x)\phi(y)$, for each $x, y \in U$ s.t. $xy \in U$, and $\phi(\tilde{x}^{-1}) = \phi(\tilde{x})^{-1}$, for each $\tilde{x} \in U$ s.t. $\tilde{x}^{-1} \in U$. In the following definition, a class of LCA groups which consists of LCA groups, which acts locally like euclidean spaces is given:

Definition 3.5 (LCA group of Lie Type). *Let $\mathcal{G}$ be an LCA group. $\mathcal{G}$ is said to be of Lie type, if $\mathcal{G}$ is locally isomorphic to $\mathbb{R}^n$, for some $n \in \mathbb{N}_0$.*

As the name asserts, LCA group of Lie type has a connection to the so-called Lie group, which is defined roughly as a group, which is a differentiable manifold, and whose group operations are compatible with its smooth structure. The theory of Lie groups has broad applications in various areas of mathematics and physics. The discussion on this connection, can be found on p. 96 in [9]. A relatively good hint on the structure of LCA groups of Lie type can be given:

Theorem 3.20 (Third Structure Theorem). *Let $\mathcal{G}$ be an LCA group of Lie type. Then $\mathcal{G}$ is homeomorphic to $\mathbb{R}^k \times \mathbb{T}^l \times D$, for some $k, l \in \mathbb{N}_0$ and discrete abelian group D.*

The following corollary is an easy application of Thm. 3.19 and Thm. 3.20, which states that compactly generated LCA groups and LCA groups of Lie type are dual to each other:

Corollary 3.21. *Let $\mathcal{G}$ be an LCA group. $\mathcal{G}$ is compactly generated if and only if $\mathcal{G}^\wedge$ is an LCA group of Lie type.*

Proof. "$\Rightarrow$": Let $\mathcal{G}$ be compactly generated. Then, by thm. 3.19, $\mathcal{G}$ is isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m \times K$, for some $n, m \in \mathbb{N}_0$, and K a compact LCA group. Applying thm. 3.7, and since $\mathbb{R}^n \cong \mathbb{Z}^n$, $\mathbb{Z}^m \cong \mathbb{T}^m$, and $K \cong D$, for a discrete LCA group D (thm. 3.6), it follows that $\mathcal{G}^\wedge \cong \mathbb{Z}^n \times \mathbb{T}^m \times D$. This shows by Thm. 3.20 that $\mathcal{G}^\wedge$ is an LCA group of Lie type.

"$\Leftarrow$": Let $\mathcal{G}$ be of Lie type. Then $\mathcal{G} \cong \mathbb{R}^k \times \mathbb{T}^l \times D$, for some $k, l \in \mathbb{N}_0$, and a discrete abelian group D. Hence by thm. 3.19 and thm. 3.6, $\mathcal{G}^{\wedge\wedge}$ is topological group isomorphic to $\mathbb{R}^k \times \mathbb{Z}^l \times K$, for a compact LCA group K. We borrow the result which shall be given later (thm 3.33), which says that $\mathcal{G}$ is isomorphic to $\mathcal{G}^{\wedge\wedge}$, to conclude the desired statement. ∎

3.5. "Dualizing" Topological Group Homomorphism and Exact Sequences

For convenient, we first give the following notion related to a subgroup:

Definition 3.6 (Annihilator). *Let $\mathcal{G}$ be an LCA group, and $H \leqslant \mathcal{G}$. The annihilator of H in $G^\wedge$ is defined as the set:*

$$\mathrm{A}_{\mathcal{G}^\wedge}(H) := \{\gamma \in \mathcal{G}^\wedge : \gamma(x) = 1, \quad \forall x \in H\}.$$

Analogously, given a subgroup $\tilde{H}$ of the dual group $\mathcal{G}^\wedge$ we define the annihilator of $\tilde{H}$ in $\mathcal{G}$ as:

$$\mathrm{A}_{\mathcal{G}}(\tilde{H}) := \{x \in \mathcal{G} : \gamma(x) = 1, \quad \forall \gamma \in \tilde{H}\}.$$

We can give the following immediate but fundamental characterization of elements of an annihilator in a dual group:

Lemma 3.22. *Let H be a subgroup of an LCA group $\mathcal{G}$. For each $\gamma \in \mathrm{A}_{\mathcal{G}^\wedge}(H)$, it holds for $x \in \mathcal{G}$:*

$$\gamma(xh) = \gamma(x), \quad \forall h \in H, \tag{3.7}$$

i.e. γ is H-periodic. Furthermore, there exists some $\tilde{\gamma} \in (\mathcal{G}/H)^\wedge$, s.t. $\gamma = \tilde{\gamma} \circ \mathrm{q}$, where $\mathrm{q} : \mathcal{G} \to \mathcal{G}/H$ denotes the canonical quotient homomorphism.

Proof. Take a $\gamma \in \mathrm{A}_{\mathcal{G}^\wedge}(H)$, and let $x \in \mathcal{G}$. Since characters are homomorphism, and by the definition of annihilator, one obtains immediately: $\gamma(xh) = \gamma(x)\gamma(h) = \gamma(x)$, $\forall h \in H$, which shows the first statement. Furthermore, since γ is constant on each cosets of H in $\mathcal{G}$, it follows that there exists a homomorphism $\tilde{\gamma} : \mathcal{G}/H \to \mathbb{C}$, for which $\gamma = \tilde{\gamma} \circ \mathrm{q}$. Since γ is continuous, and q is open, it must hold that $\tilde{\gamma}$ is continuous, which shows the second statement. ∎

Given a continuous homomorphism between LCA groups $f : \mathcal{G} \to \tilde{\mathcal{G}}$. One can define a mapping canonical to f which assign a character on $\tilde{\mathcal{G}}$ to a character on $\mathcal{G}$ as follows:

$$f^\wedge(\gamma) = \gamma \circ f, \quad \forall \gamma \in \tilde{\mathcal{G}}^\wedge. \tag{3.8}$$

We give in the following lemma some properties of this canonical mapping, in particular that of the canonical quotient mapping, which is useful for our later approach:

Lemma 3.23. *Let $\mathcal{G}$ and $\tilde{\mathcal{G}}$ be LCA groups, and $f : \mathcal{G} \to \tilde{\mathcal{G}}$ be a continuous homomorphism. Then the following statements holds:*

(a) $f^\wedge$ is a continuous homomorphism between $\tilde{\mathcal{G}}^\wedge$ to $\mathcal{G}^\wedge$.

(b) If f is surjective, then $f^\wedge$ is injective.

(c) If f is surjective, s.t. for each compact subset $\tilde{K}$ of $\tilde{\mathcal{G}}$, there exists a compact subset K of $\mathcal{G}$ s.t. $\tilde{K}$ is the image of K under f, then $f^\wedge$ is an embedding. So, $f^\wedge : \tilde{\mathcal{G}}^\wedge \to f^\wedge(\tilde{\mathcal{G}}^\wedge)$ is a topological group isomorphism.

(d) Let $\tilde{G} = G/H$, where $H \leqslant G$ is a closed subgroup. If f is a quotient map, then $f^\wedge$ is an embedding. Again, $f^\wedge : (\mathcal{G}/H)^\wedge \to f^\wedge((\mathcal{G}/H)^\wedge)$ is a topological group isomorphism.

(e) If f is injective and open, then $f^\wedge$ is surjective.

(f) If f is a topological group isomorphism, then $f^\wedge$ is also a topological group isomorphism.

Proof. (a): It is straightforward to show that $\forall \gamma \in \tilde{\mathcal{G}}^\wedge$, $f^\wedge(\gamma)$ is a character on $\mathcal{G}$, and that $f^\wedge$ preserves the group operation. So hence one obtains that $f^\wedge$ is a homomorphism. It is obvious that $f^\wedge$ is a homomorphism. To proof the continuity of $f^\wedge$, it clearly suffices to show the continuity of $f^\wedge$ at the identity $1_{\tilde{\mathcal{G}}^\wedge}$ of $\tilde{\mathcal{G}}^\wedge$. Let $K \subseteq \tilde{\mathcal{G}}$ be compact. Take the neighborhood $\mathcal{W}$ of the identity $1_{\tilde{\mathcal{G}}^\wedge}$, where $\mathcal{W}$ is of the form $\mathcal{W}(f(K), U)$, with U is an open neighborhood of the identity in $\mathbb{T}$. Since f is continuous, it follows immediately that $f(K)$ is compact. Finally, observe that $f^\wedge(W) \subseteq \mathcal{W}_{\mathcal{G}^\wedge}(K, U)$. So, we obtain the statement (a).

(b): Now, let f be surjective, and assume that $f^\wedge(\gamma_1) = f^\wedge(\gamma_2)$ for some $\gamma_1, \gamma_2 \in \tilde{\mathcal{G}}^\wedge$. By reformulation, it follows: $(\gamma_1 \circ f)(x) = (\gamma_2 \circ f)(x)$, $\forall x \in \mathcal{G}$. Since f is surjective, previous equality is equivalent to $\gamma_1(x') = \gamma_2(x')$, $\forall x' \in \tilde{\mathcal{G}}$, which shows that $\gamma_1 = \gamma_2$. Hence, $f^\wedge$ is injective, and the statement (b) is shown.

(c): let f fulfilled the required properties. From (b), we already know that $f^\wedge$ is injective. So, it remains to show that $f^\wedge$ is an open map, or equivalently, the image of each element of the open neighborhood base of identity in $\tilde{\mathcal{G}}$ is an element of open neighborhood base of the identity of $\mathcal{G}^\wedge$ relative to ran(f). Take a compact subset $\tilde{K} \subseteq \tilde{\mathcal{G}}$, and the corresponding compact subset $K \subseteq \mathcal{G}$, s.t. $\tilde{K} = f(K)$, and take a neighborhood U of the identity in $\mathbb{T}$. $\mathcal{W}_{\tilde{\mathcal{G}}^\wedge}(\tilde{K}, U)$ is clearly an element of the neighborhood base of the identity in $\tilde{\mathcal{G}}^\wedge$ w.r.t. the compact open topology. Take $\gamma \in \mathcal{W}_{\tilde{\mathcal{G}}^\wedge}(\tilde{K}, U)$. Since $f^\wedge(\gamma) = \gamma \circ f$, and $\tilde{K} = f(K)$, it follows immediately that $f^\wedge(\gamma)(K) = \gamma(\tilde{K}) \subseteq U$. Hence $f^\wedge(\gamma) \in \mathcal{W}_{\mathcal{G}^\wedge}(K, U)$, and since

$f^\wedge$ is injective, we obtain as desired: $f^\wedge(\mathcal{W}_{\tilde{\mathcal{G}}^\wedge}(\tilde{K}, U)) = \operatorname{ran}(f^\wedge) \cap \mathcal{W}_{\mathcal{G}^\wedge}(K, U)$. Finally the remaining statement follows from (a)

(d): notice that from the statement (b) in proposition (C.10), and from the surjectivity of quotient map, the condition required in statement (c) is fulfilled. So, one obtains immediately the desired statement.

(e): See f as the continuous (algebraic) isomorphism $f : \mathcal{G} \to f(\mathcal{G})$. Furthermore, notice that $f(\mathcal{G}) \leqslant \tilde{\mathcal{G}}$. Clearly, $f^{-1} : f(\mathcal{G}) \to \mathcal{G}$ is also an (algebraic) isomorphism but not necessarily continuous. For a $\gamma \in \mathcal{G}^\wedge$, define the algebraic homomorphism $\tilde{\gamma} : f(\mathcal{G}) \to \mathbb{T}$ by $\tilde{\gamma} := \gamma \circ f^{-1}$, which is again not necessarily continuous. Lemma 3.13 asserts that $\tilde{\gamma}$ can be extended to a (not necessarily continuous) homomorphism ξ between $\tilde{\mathcal{G}}$ and $\mathbb{T}$. In other words, $\gamma = \xi \circ f$, where f is seen as $f : \mathcal{G} \to \tilde{\mathcal{G}}$. Furthermore, since f is open, and γ is continuous, it follows that ξ is also continuous. Summarized, for each $\gamma \in \mathcal{G}^\wedge$, there exists a $\xi \in \tilde{\mathcal{G}}^\wedge$ s.t. $\gamma = \xi \circ f = f^\wedge(\xi)$, as desired.

Statement (f) should be obvious, else, (f) follows from (b), (e), (c). ∎

Acoordingly, some statements concerning to the "dual" of the inclusion mapping and another statement concerning to the "dual" of quotient mapping can be made:

Lemma 3.24. *Let $\mathcal{G}$ be an LCA group, and $H \leqslant \mathcal{G}$. Furthermore, let inc : $H \to \mathcal{G}$ and $\mathrm{q} : \mathcal{G} \to \mathcal{G}/H$ be canonical inclusion, and canonical quotient mapping, respectively. Then the following holds:*

(a) $\operatorname{ran} \mathrm{q}^\wedge = \ker i^\wedge = \mathrm{A}_{\mathcal{G}^\wedge}(H)$. In particular, $(\mathcal{G}/H^\wedge)^\wedge$ is topological group isomorphic to $\mathrm{A}_{\mathcal{G}^\wedge}(H)$ by $\mathrm{q}^\wedge$.

(b) If H is compact, then inc$^\wedge$ is open and surjective.

Proof. (a): It is not hard to see that:

$$\mathrm{inc}^\wedge \circ \mathrm{q}^\wedge = (\mathrm{q} \circ \mathrm{inc})^\wedge. \tag{3.9}$$

Indeed, let $\gamma \in (\mathcal{G}/H)^\wedge$. By some elementary computations: $(\mathrm{inc}^\wedge \circ \mathrm{q}^\wedge)(\gamma) = \mathrm{inc}^\wedge(\gamma \circ \mathrm{q}) = \gamma \circ \mathrm{q} \circ \mathrm{inc}$, which shows the claim. Furthermore, Notice that since $\operatorname{ran}(\mathrm{inc}) = H$ and $\ker(\mathrm{q}) = H$, it follows immediately that $\operatorname{ran}((\mathrm{q} \circ \mathrm{inc})^\wedge) = \{1_{(\mathcal{G}/H)^\wedge}\}$, and correspondingly from (3.9), $\operatorname{ran}(\mathrm{inc}^\wedge \circ \mathrm{q}^\wedge) = \{1_{(\mathcal{G}/H)^\wedge}\}$. From the latter, it follows: $\mathrm{inc}^\wedge(\gamma) = \{1_{H^\wedge}\}$, $\forall \gamma \in \operatorname{ran}(\mathrm{q}^\wedge)$, which shows that $\operatorname{ran}(\mathrm{q}^\wedge) \subseteq \ker(\mathrm{inc}^\wedge)$. Now, Notice that $\ker(\mathrm{inc}^\wedge) := \{\gamma \in \mathcal{G}^\wedge : \gamma(x) = 1_{\mathbb{T}}, \forall x \in H\} = \mathrm{A}_{\mathcal{G}^\wedge}(H)$. So, from lemma 3.22, it follows that for a $\gamma \in \ker(\mathrm{inc}^\wedge)$, there exists a $\tilde{\gamma} \in (\mathcal{G}/H)^\wedge$, for which $\gamma = \tilde{\gamma} \circ \mathrm{q} = \mathrm{q}^\wedge(\tilde{\gamma})$, which shows that $\ker(\mathrm{inc}^\wedge) \subseteq \operatorname{ran}(\mathrm{q}^\wedge)$. Summarily, we have $\operatorname{ran}(\mathrm{q}^\wedge) = \ker(\mathrm{inc}^\wedge) = \mathrm{A}_{G^\wedge}(H)$. Finally, The statement (d) in lemma 3.23 asserts that $\mathrm{q}^\wedge$ is topological group isomorphism between $(\mathcal{G}/\mathcal{H})^\wedge$ and $\mathrm{q}^\wedge((\mathcal{G}/H)^\wedge) = \mathrm{A}_{\mathcal{G}^\wedge}(H)$.

(b): To show that inc$^\wedge$ is surjective, it is clear, that we have to show: for each $\gamma \in K^\wedge$, there exists $\tilde{\gamma} \in \mathcal{G}^\wedge$ s.t. $\gamma = \mathrm{inc}^\wedge(\tilde{\gamma}) = \tilde{\gamma}|_K$. Corollary 3.11 provides the required statement. By noticing that $H^\wedge$ is discrete, since H is compact, and from the fact that every function mapping to a discrete space is open, it follows immediately that inc$^\wedge$ is open. ∎

Now, we are able to give a generalization of previous idea:

Lemma 3.25. *Let $\mathcal{G}$ and $\mathcal{G}_1$ be LCA groups, and $f : \mathcal{G} \to \mathcal{G}_1$ be an open continuous homomorphism. Then the following holds:*

(a) Let $\mathcal{G}$ be compact. If f is injective, then $f^\wedge$ is an open and surjective.

(b) If f is surjective, then $f^\wedge$ is an embedding. Furthermore, ran$(f^\wedge) =$ ran$(\mathrm{q}^\wedge)$.

Proof. (a): If f is injective, then it can obviously be written as $f = \mathrm{inc} \circ \tilde{f}$, where $\tilde{f} = f : \mathcal{G} \to f(\mathcal{G})$, and inc : $f(\mathcal{G}) \to \mathcal{G}_1$ the canonical inclusion mapping. Notice that $f^\wedge = \tilde{f}^\wedge \circ \mathrm{inc}^\wedge$. Now, let $\mathcal{G}_1$ be compact, Applying item (b) in lemma 3.24, one immediately see that $f^\wedge$ is the composition of two surjective and open mapping ($\tilde{f}$ is a isomorphism), and hence also surjective and open.

(b): By the assumption on f, it follows that f is "quotient-like", i.e. $f = \tilde{f} \circ \mathrm{q}$, where $\mathrm{q} : \mathcal{G} \to \mathcal{G}/\ker f$, and $\tilde{f} : \mathcal{G}/\ker f \to \mathcal{G}_1$ are canonical quotient homomorphism, and - top. group isomorphism, respectively. From item (a) in 3.24, it follows that $\mathrm{q}^\wedge$ is an embedding of topological groups, and from (f) in lemma 3.23, $\tilde{f}^\wedge$ is a top. group isomorphism. Hence, $f^\wedge = \mathrm{q}^\wedge \circ \tilde{f}^\wedge$ is a top. group embedding, and $\mathrm{ran}(f^\wedge) = \mathrm{ran}(\mathrm{q}^\wedge)$.

For convenient, more insight to the connection of different structures, and in order to be able to use a fundamental lemma in category theory for later purposes, we use the following term, which is adjusted to the theory of LCA groups, called short exact sequence. Let $\mathcal{G}$, $\mathcal{G}_1$, and $\mathcal{G}_2$ be LCA groups, which we write as usual multiplicatively, and let $f_1 : \mathcal{G}_1 \to \mathcal{G}_2$, and $f_2 : \mathcal{G}_2 \to \mathcal{G}_3$ are continuous homomorphism. Consider the following, called short sequence:

$$\{1\} \to \mathcal{G}_1 \xrightarrow{f_1} \mathcal{G}_2 \xrightarrow{f_2} \mathcal{G}_3 \to \{1\}. \tag{3.10}$$

Above short sequence is said to be exact if f_1 is injective, f_2 is surjective, and $\mathrm{ran}\, f_1 = \ker f_2$. For purposes of the analysis of topological groups, it is helpful to consider a stronger form of exact sequences. We say that the exact sequence (3.10) is proper, if f_1 and f_2 are proper, or in a more convenient term, are open. If (3.10) is a proper exact sequence, one can immediately conclude that f_1 is a monomorphism between topological groups, or equivalently, $f_1 : \mathcal{G}_1 \to f_1(\mathcal{G}_1)$ is a topological group isomorphism. Furthermore, in this case, one can also conclude, that the canonical continuous homomorphism $\tilde{f}_2 : \mathcal{G}_2/\ker f_2 \to \mathcal{G}_3$ is a topological group isomorphism.

For convenient, we apply previous language in the following:

Examples 3.26. *Let $\mathcal{G}$ be an LCA group, and $H \leqslant \mathcal{G}$ closed. That inc and q are continuous homomorphism is clear. Furthermore, one see immediately, that inc is injective, q is surjective, and $\mathrm{ran\,inc} = \ker \mathrm{q} = H$. So, the fact that the sequence is exact is clear. Notice that the inclusion - and quotient mapping are always open. Summarily, the following sequence is proper exact:*

$$\{1\} \to H \xrightarrow{inc} \mathcal{G} \xrightarrow{\mathrm{q}} \mathcal{G}/H \to \{1\}.$$

Corollary 3.27. *Let $\mathcal{G}$, $\mathcal{G}_1$, and $\mathcal{G}_2$ be LCA groups. If the following short sequence is proper exact:*

$$\{1\} \to \mathcal{G}_1 \xrightarrow{f_1} \mathcal{G} \xrightarrow{f_2} \mathcal{G}_2 \to \{1\}, \tag{3.11}$$

, then the "dual" short sequence:

$$\{1\} \to \mathcal{G}_2^\wedge \xrightarrow{f_2^\wedge} \mathcal{G} \xrightarrow{f_1^\wedge} \mathcal{G}_1^\wedge \to \{1\}, \tag{3.12}$$

is also exact, with $f_2^\wedge$ an embedding.

Furthermore, if in addition $\mathcal{G}_1$ is compact, then 3.12 is proper exact.

Proof. We already see that $f_1^\wedge$ and $f_2^\wedge$ are continuous homomorphism. From the proof of lemma 3.25, it follows that $\ker(f_1^\wedge) = \ker(\mathrm{inc}^\wedge)$, and $\mathrm{ran}(f_2^\wedge) = \mathrm{ran}(\mathrm{q}^\wedge)$, where $\mathrm{inc} : f_1(\mathcal{G}_1) \to \mathcal{G}$ the canonical inclusion, and $\mathrm{q} : \mathcal{G} \to \mathcal{G}/\ker f_2$ the canonical quotient mapping. Furthermore, from exactness assumption of (3.11), it follows that $f_1(\mathcal{G}_1) = \ker f_2$. Hence, by lemma 3.24, it follows that $\ker \mathrm{inc}^\wedge = \mathrm{ran}(\mathrm{q}^\wedge)$, and correspondingly, $\mathrm{ran}\, f_2^\wedge = \ker f_1^\wedge$. So it remains now to show that $f_1^\wedge$ is injective, and $f_2^\wedge$ is surjective. Item (b) in 3.25, and item (e) in lemma 3.23 provide the desired statement

For the case that $\mathcal{G}_1$ is compact, apply Lemma 3.25, to conclude that $f_2^\wedge$ is open. ■

In the next section we consider with the so-called Pontryagin duality. Roughly speaking, this principle says that the dual of the dual group of an LCA group is simply the LCA group itself.

3.6. Pontryagin Duality

Consider an LCA group $\mathcal{G}$. Recall that, with $\mathcal{G}^{\wedge\wedge}$, it is meant the set of all continuous homomorphism from $\mathcal{G}^{\wedge}$ to $\mathbb{T}$. Consider the map $\alpha_{\mathcal{G}}$, which assigns each $x \in \mathcal{G}$ a character (not yet shown) on $\mathcal{G}^{\wedge}$ by $\alpha_{\mathcal{G}}(x)(\gamma) = \gamma(x)$, $\forall \gamma \in \mathcal{G}^{\wedge}$. In this section, we will see that $\alpha_{\mathcal{G}}$ is a natural identification, i.e. a topological group isomorphism, between $\mathcal{G}$ and $\mathcal{G}^{\wedge\wedge}$.

It is not hard to establish the fact, that $\alpha_{\mathcal{G}}$ is a continuous homomorphism between $\mathcal{G}$ and $\mathcal{G}^{\wedge}$, as done in the following lemma:

Lemma 3.28. *Let $\mathcal{G}$ be an LCA group, then the mapping $\alpha_{\mathcal{G}}$ is a continuous homomorphism between $\mathcal{G}$ and $\mathcal{G}^{\wedge\wedge}$.*

Proof. It is straightforward to see that for $x \in \mathcal{G}$, $\alpha_{\mathcal{G}}(x)$ is a homomorphism from $\mathcal{G}^{\wedge}$ to $\mathbb{T}$:

$$\alpha_{\mathcal{G}}(x)(\gamma\gamma^{'}) = (\gamma\gamma^{'})(x) = \gamma(x)\gamma^{'}(x) = \alpha_{\mathcal{G}}(x)(\gamma) \cdot \alpha_{\mathcal{G}}(x)(\gamma^{'}), \quad \forall \gamma, \gamma^{'} \in \mathcal{G}^{\wedge}.$$

Now, we show that for $x \in \mathcal{G}$, $\alpha_{\mathcal{G}}(x)$ is continuous in $\mathcal{G}^{\wedge}$, and accordingly a character for $\mathcal{G}^{\wedge}$. Since $\mathcal{G}^{\wedge}$ is an LCA group, it suffices to show that $\alpha_{\mathcal{G}}(x)$ is continuous at the identity $1_{\mathcal{G}^{\wedge}}$. For a neighborhood U of the identity 1 in $\mathbb{T}$, choose the neighborhood $\mathcal{W}(\{x\}, U)$ of $1_{\mathcal{G}^{\wedge}}$, and notice that $\alpha_{\mathcal{G}}(x)(\mathcal{W}(\{x\}, U)) \subseteq U$, which shows the continuity of $\alpha_{\mathcal{G}}(x)$ at $1_{\mathcal{G}^{\wedge}}$.

It is not hard to see that $\alpha_{\mathcal{G}}$ is a homomorphism. Indeed, for $x, y \in \gamma$: $\alpha_{\mathcal{G}}(xy)(\gamma) = \gamma(xy) = \gamma(x)\gamma(y) = \alpha_{\mathcal{G}}(x)(\gamma) \cdot \alpha_{\mathcal{G}}(y)(\gamma)$, $\forall \gamma \in \mathcal{G}^{\wedge}$.

It remains to show that $\alpha_{\mathcal{G}}$ is continuous. Since $\mathcal{G}^{\wedge\wedge}$ is an LCA group, we only need to show that $\alpha_{\mathcal{G}}$ is continuous at the identity $1_{\mathcal{G}}$ of $\mathcal{G}$. Let $\mathcal{W}_{\mathcal{G}^{\wedge\wedge}}(K_{\mathcal{G}^{\wedge}}, U)$ be an open basic neighborhood of $\alpha_{\mathcal{G}}(1_{\mathcal{G}}) = 1_{\mathcal{G}^{\wedge}}$, where $K_{\mathcal{G}^{\wedge}} \subseteq \mathcal{G}^{\wedge}$ is compact and non-empty, and U is an open neighborhood of $1_{\mathbb{T}}$. Now, we construct an open neighborhood V of $1_{\mathcal{G}}$ s.t. $\alpha_{\mathcal{G}}(V) \subseteq \mathcal{W}_{\mathcal{G}^{\wedge\wedge}}(K_{\mathcal{G}^{\wedge}}, U)$ as follows:

Take a relatively compact open neighborhood $A_{\mathcal{G}}$ of $1_{\mathcal{G}}$, and an open symmetric neighborhood B of $1_{\mathbb{T}}$ s.t. $B^2 \subseteq U$. Clearly by this choice, $\mathcal{W}_{\mathcal{G}^{\wedge}}(\overline{A_{\mathcal{G}}}, B)$ is an neighborhood of $1_{\mathcal{G}^{\wedge}}$, and this set is in particular open in $\mathcal{G}^{\wedge}$. Since $K_{\mathcal{G}^{\wedge}}$ is compact in $\mathcal{G}^{\wedge}$, it follows immediately that $K_{\mathcal{G}^{\wedge}}$ can be covered by finitely many open sets in $\mathcal{G}^{\wedge}$. So, there exists finitely many collection of characters $\gamma_1, \ldots, \gamma_n$ on $\mathcal{G}^{\wedge}$, s.t. $K_{\mathcal{G}^{\wedge}} \subseteq \bigcup_{i \in [n]} \gamma_i \mathcal{W}_{\mathcal{G}^{\wedge}}(\overline{A_{\mathcal{G}}}, B)$. Notice that by the continuity of characters at the identity of $\mathcal{G}$, there exists for each $i \in [n]$, $V_i \subseteq \mathcal{G}$, s.t. $\gamma_i(V_i) \subseteq B$, and $V_i \subseteq U$. So we define the desired set $V \subseteq \mathcal{G}$ as $V := U \cap V_1 \cap \ldots \cap V_n$, which is clearly open. Notice that, since $V \subseteq V_i$, $\forall i \in [n]$, it follows immediately that $\gamma_i(V) \subseteq B$, $\forall i \in [n]$.

Let be $\gamma \in K_{\mathcal{G}^{\wedge}}$. From the finite covering property of $K_{\mathcal{G}^{\wedge}}$ by open sets $\gamma_i \mathcal{W}_{\mathcal{G}^{\wedge}}(\overline{A_{\mathcal{G}}}, B)$, $i \in [n]$, it follows that there exists $i_0 \in [n]$, s.t. $\gamma \in \gamma_{i_0} \mathcal{W}_{\mathcal{G}^{\wedge}}(\overline{A_{\mathcal{G}}}, B)$. In particular, γ can be written as $\gamma = \gamma_{i_0}\xi$, for some $\xi \in \mathcal{W}_{\mathcal{G}^{\wedge}}(\overline{A_{\mathcal{G}}}, B)$. So for all $x \in V$, it follows $\gamma(x) = \gamma_{i_0}(x)\xi(x) \in B \cdot B \subseteq V$. Hence, $\forall x \in V$ $\alpha_{\mathcal{G}}(x)(\gamma) = \gamma(x) \in V$, $\forall \gamma \in K_{\mathcal{G}^{\wedge}}$, and equivalently $\alpha_{\mathcal{G}}(V) \subseteq \mathcal{W}_{\mathcal{G}^{\wedge\wedge}}(K_{\mathcal{G}^{\wedge}}, U)$ as desired. ∎

From the discussions made in previous section, one can be sure that the following property of the natural mapping $\alpha_{\mathcal{G}}$ holds:

Corollary 3.29. *Let $\mathcal{G}$ be an LCA group. Then the natural mapping $\alpha_{\mathcal{G}}$ is a monomorphism between $\mathcal{G}$ and $\mathcal{G}^{\wedge\wedge}$. Furthermore, $\alpha_{\mathcal{G}}(\mathcal{G})$ is a separating set of $\mathcal{G}^{\wedge}$.*

Proof. For the first statement: We already know from lemma 3.28 that $\alpha_{\mathcal{G}}$ is a homomorphism. So, it remains to show that $\alpha_{\mathcal{G}}$ is injective. From the separation property of $\mathcal{G}^{\wedge}$ (thm. 3.14), it follows that for each $x \in \mathcal{G}$, there exists always a character γ on $\mathcal{G}$ s.t. $\gamma(x) \neq 1_{\mathbb{T}}$. Hence the kernel of $\alpha_{\mathcal{G}}$, $\ker(\alpha_{\mathcal{G}}) = \{x \in \mathcal{G} : \gamma(x) = 1_{\mathbb{T}},\ \forall \gamma \in \mathcal{G}^{\wedge}\}$, contains only $1_{\mathcal{G}}$, which shows that $\alpha_{\mathcal{G}}$ is injective.

For the second statement. From separating property of $\mathcal{G}^{\wedge}$. For $x \in \mathcal{G}$, $\alpha_{\mathcal{G}}(x)(\gamma) = \gamma(x)$ ∎

For products of LCA groups, we have the following statement

Lemma 3.30. *Let $\{\mathcal{G}_j\}_{j\in[N]}$ be a finite collection of LCA groups, and $\{\alpha_{\mathcal{G}_j}\}_{j\in[N]}$ be the corresponding collection of natural mapping. Denote the product of $\{\mathcal{G}_j\}_{j\in[N]}$ by $\mathcal{G}$. Suppose that for each $j \in [N]$, $\mathcal{G}_j$ is identifiable with $\mathcal{G}_j^{\wedge\wedge}$ by the corresponding natural mapping $\alpha_{\mathcal{G}_j}$. Then $\mathcal{G}$ is identifiable with $\mathcal{G}^{\wedge\wedge}$.*

Proof. Apply thm. 3.7 2 times , to obtain an isomorphism $\mathcal{G}^{\wedge\wedge}$ and $\prod_j \mathcal{G}_j^{\wedge\wedge}$. Construct by $\{\alpha_{\mathcal{G}_j}\}_j$ the mapping $\phi : \prod_j \mathcal{G}_j \to \prod_j \mathcal{G}_j^{\wedge\wedge}$. It is not hard to see, that since each $\alpha_{\mathcal{G}_j}$, $j \in [N]$ is an isomorphism, it follows that the product of isomorphic mapping ϕ is also an isomorphism. Hence the statement holds. ∎

It needs some efforts to establish the fact that the mapping α is surjective and open. The fact that $\mathcal{G}$ is either compact or elementary is fairly easy to show:

Lemma 3.31. *Let $\mathcal{G}$ be an LCA group. If $\mathcal{G}$ is either discrete, or compact, or elementary, then $\alpha_{\mathcal{G}}$ is a topological group isomorphism*

Sketch of Proof. From corollary 3.29, we already know that $\alpha_{\mathcal{G}}$ is a continuous injective homomorphism. So, it remains to show that $\alpha_{\mathcal{G}}$ is surjective and open.

If $\mathcal{G}$ is discrete, then $\mathcal{G}^{\wedge}$ is obviously compact. We already know that $\alpha_{\mathcal{G}}(\mathcal{G}^{\wedge})$ separates points of $\mathcal{G}^{\wedge}$. Furthermore, $\alpha_{\mathcal{G}}(\mathcal{G}^{\wedge}) \leqslant \mathcal{G}$, is compact, as it is an image of a compact set under continuous mapping. So, from lemma 3.10, it follows immediately that $\alpha_{\mathcal{G}}(\mathcal{G}) = \mathcal{G}^{\wedge\wedge}$. Furthermore, since $\mathcal{G}^{\wedge\wedge}$ is discrete, and any mapping to a discrete space is open, $\alpha_{\mathcal{G}}$ is open, as desired.

If $\mathcal{G}$ is compact, it follows immediately that $\mathcal{G}^{\wedge}$ is discrete. Notice that $\alpha_{\mathcal{G}}(\mathcal{G})$ separates points of $\mathcal{G}^{\wedge}$. Hence $\alpha_{\mathcal{G}}(\mathcal{G})$ is dense in $\mathcal{G}^{\wedge\wedge}$ by corollary 3.12. Furthermore, $\alpha_{\mathcal{G}}(\mathcal{G}^{\wedge}) \leqslant \mathcal{G}$, is compact, as it is an image of a compact set under continuous mapping. In particular, $\alpha_{\mathcal{G}}(\mathcal{G})$ is closed in $\mathcal{G}^{\wedge\wedge}$, and hence $\alpha_{\mathcal{G}}(\mathcal{G}) = \overline{\alpha_{\mathcal{G}}(\mathcal{G})} = \mathcal{G}^{\wedge\wedge}$. From the fact that $\mathcal{G}$ is compact, and by Open Mapping Theorem of topological groups, it follows that $\alpha_{\mathcal{G}}$ is open.

Now, in case that $\mathcal{G}$ is elementary, from lemma 3.30 and above discussions. It is sufficient to show that $\alpha_{\mathbb{R}}$ is open and surjective, which is a fairly easy task. ∎

Now, we are able to sketch the proof of Pontryagin-van Kampen duality for the case that $\mathcal{G}$ is compactly generated:

Lemma 3.32. *Let $\mathcal{G}$ be a compactly generated LCA group. Then the map $\alpha_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}^{\wedge\wedge}$ is an isomorphism.*

Sketch of Proof. By thm. 3.17, we already see that $\mathcal{G}$ possessess a compact subgroup K s.t. $\mathcal{G}/K$ is topological group isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{T}^l \times D$, where $n, m, l \in \mathbb{N}_0$, and D a finite discrete abelian group. Furthermore, we have a proper exact sequence:

$$\{1\} \to K \xrightarrow{\text{inc}} \mathcal{G} \xrightarrow{\text{q}} \mathcal{G}/K \to \{1\},$$

where inc is inclusion mapping, and q is the canonical quotient mapping. Hence, applying corollary 3.27 to previous exact sequence, it yields that the following sequence is also proper exact:

$$\{1\} \to (\mathcal{G}/K)^{\wedge} \xrightarrow{\text{q}^{\wedge}} \mathcal{G}^{\wedge} \xrightarrow{\text{inc}^{\wedge}} K^{\wedge} \to \{1\}.$$

Subsequently, applying again corollary 3.27 to above proper exact short sequence, it follows that the following short sequence is also exact[2]:

$$\{1\} \to K^{\wedge\wedge} \xrightarrow{\text{inc}^{\wedge\wedge}} \mathcal{G}^{\wedge\wedge} \xrightarrow{\text{q}^{\wedge\wedge}} (\mathcal{G}/K)^{\wedge\wedge} \to \{1\},$$

[2]But not necessarily proper. However, this suffices for our purpose. By different technique, one can show that the resulted exact sequence is also proper see e.g. Prop. 36 and Thm. 22 in [42]

Now consider the following diagram:

$$\begin{tikzcd}
\{1\} \arrow[r] & K \arrow[r, "\mathrm{inc}"] \arrow[d, "\alpha_K"'] & \mathcal{G} \arrow[r, "\mathrm{q}"] \arrow[d, "\alpha_{\mathcal{G}}"'] & \mathcal{G}/K \arrow[r] \arrow[d, "\alpha_{\mathcal{G}/K}"'] & \{1\} \\
\{1\} \arrow[r] & K^{\wedge\wedge} \arrow[r, "\mathrm{inc}^{\wedge\wedge}"] & \mathcal{G}^{\wedge\wedge} \arrow[r, "\mathrm{q}^{\wedge\wedge}"] & (\mathcal{G}/K)^{\wedge\wedge} \arrow[r] & \{1\}
\end{tikzcd}$$

It is not hard to verify that above diagram commutes, i.e. $\mathrm{inc}^{\wedge\wedge} \circ \alpha_K = \alpha_{\mathcal{G}} \circ \mathrm{inc}$, and $\mathrm{q}^{\wedge\wedge} \circ \alpha_{\mathcal{G}} = \alpha_{\mathcal{G}/K} \circ \mathrm{q}$. Furthermore, lemma 3.31 asserts that α_K and $\alpha_{\mathcal{G}/K}$ are topological group isomorphism, and in particular bijective (or algebraic isomorphism). Since the horizontal sequences are exact, and α_K and $\alpha_{\mathcal{G}/K}$ are algebraic isomorphism, it follows from Five-Lemma of category theory (with category of Abelian groups), that $\alpha_{\mathcal{G}}$ is also bijective. Since we have seen that α is continuous and $\mathcal{G}$ is compactly generated, it follows from Open Mapping Theorem of topological groups, that $\alpha_{\mathcal{G}}$ is open. So the statement is obtained. ∎

Since we shall later concentrate ourself to compactly generated LCA groups, it is not necessary to show the Pontryagin-van Kampen duality for general LCA groups. But, for sake of completeness, we state it in the following:

Theorem 3.33 (Pontryagin-van Kampen Duality Theorem). *Let $\mathcal{G}$ be an LCA group. Then $\alpha_{\mathcal{G}}$ is a topological group isomorphism of $\mathcal{G}$ onto $\hat{\hat{\mathcal{G}}}$.*

Proof. Since we have shown the Pontryagin-van Kampen duality for compactly generated group, it is not a long way to show the general case. For detailed explanation, see e.g. Theorem 23 in [42]. ∎

Examples 3.34. *Consider the additive LCA group $\mathbb{R}$. We already know that the dual group of $\mathbb{R}$ is homeomorphic to real by the identification $((\cdot), \gamma_y) = e^{2\pi i(\cdot)y} \leftrightarrow y$, $y \in \mathbb{R}$.*

3.7. Dual Goups of Subgroups and Quotient Groups - Consequences of Pontryagin-van Kampen Duality

From the discussions made previously, we can infer the following characterization of annihilators:

Lemma 3.35. *Let $\mathcal{G}$ be an LCA group, and $B \leqslant \mathcal{G}$ a subgroup. Then $\mathrm{A}_{\mathcal{G}^\wedge}(B)$ is a closed subgroup of $\mathcal{G}^\wedge$.*

Proof. Notice the following reformulations of the annihilator of B in $\hat{\mathcal{G}}$:

$$\mathrm{A}_{\hat{\mathcal{G}}}(B) = \bigcap_{x \in B} \{\gamma \in \mathcal{G}^\wedge : \gamma(x) = 1\} = \bigcap_{x \in B} \ker(\alpha_{\mathcal{G}}(x)),$$

where $\alpha_{\mathcal{G}}$ is the natural identification between $\mathcal{G}$ and $\mathcal{G}^{\wedge\wedge}$. Clearly, for all $x \in M$, $\ker(\alpha_{\mathcal{G}}(x))$ is a closed subgroup of $\mathcal{G}^\wedge$, and arbitrary intersection of closed sets is closed. Hence, $\mathrm{A}_{\mathcal{G}^\wedge}(B)$ is closed. ∎

Another important facts relating to annihilators, dual group of a quotient group, and the dual group of a subgroup, is summarized in the following theorem:

Theorem 3.36. *Let $\mathcal{G}$ be an LCA group, and let H be a closed subgroup of $\mathcal{G}$, then the following holds:*

(a) *$(\mathcal{G}/H)^\wedge$ is topological group isomorphic to $\mathrm{A}_{\mathcal{G}^\wedge}(H)$ by the identification $\mathrm{A}_{\mathcal{G}^\wedge}(H) \ni \gamma \mapsto \tilde{\gamma}(\in \mathcal{G}/H)^\wedge$, where $\tilde{\gamma}([x]) = \gamma(x)$, $\forall x \in \mathcal{G}$.*

(b) $\mathrm{A}_{\mathcal{G}}(\mathrm{A}_{\mathcal{G}^\wedge}(H)) = H$.

(c) $\mathrm{A}_{\mathcal{G}^{\wedge\wedge}}(H)$ *can be identified with* $\mathrm{A}_{\mathcal{G}}(H)$ *by the identification* $\mathrm{A}_{\mathcal{G}^{\wedge\wedge}}(H) \ni \xi_x \leftrightarrow x \in \mathrm{A}_{\mathcal{G}}(K)$*, where* $\xi_x := \alpha_{\mathcal{G}}(x)$*.*

(d) $H^\wedge$ *is topological group isomorphic to* $\hat{\mathcal{G}}/A_{\mathcal{G}^\wedge}(H)$*.*

Proof. Statement (a) was already given implicitly in item (a) in 3.24

To proof (b), we first claim that:

$$\forall x \in \mathcal{G}\backslash\mathcal{H} : \exists \gamma \in \mathrm{A}_{\mathcal{G}^\wedge}(H) : \quad \gamma(x) \neq 1_{\mathbb{T}}. \tag{3.13}$$

■

Consider the topological isomorphism $\phi : (\mathcal{G}/H)^\wedge \to \mathrm{A}_{\mathcal{G}^\wedge}(H)$ given in (a). Since $\mathcal{G}/H$ is locally compact, it follows immediately from the separating property of LCA groups, that for $x \in \mathcal{G}/H$, $x \neq 1_{\mathcal{G}/H}$, there exists a character $\tilde{\gamma}$ on $\mathcal{G}/H$ for which $\tilde{\gamma}(x) \neq 1_{\mathbb{T}}$ holds, and clearly this holds on coset xH of x, i.e. $\tilde{\gamma}(xH) \neq 1_{\mathbb{T}}$. So we define $\rho = \phi(\tilde{\gamma}) \in \mathrm{A}_{\mathcal{G}^\wedge}(H)$. Hence the claim holds. This claim shows that $\mathrm{A}_{\mathcal{G}}(\mathrm{A}_{\mathcal{G}^\wedge}(H))$ is non-empty. Accordingly from its definition, it holds $\mathrm{A}_{\mathcal{G}}(\mathrm{A}_{\mathcal{G}^\wedge}(H)) = H$.

(c) Take an $\xi \in \mathrm{A}_{\mathcal{G}^{\wedge\wedge}}(H)$. From Pontryagin duality, it follows that there exists a unique $x \in \mathcal{G}$ s.t. $\alpha_{\mathcal{G}}(x) = \xi$. Furthermore, the identification is of group theoretic - and topological nature. We write ξ_x to emphasize the connection between ξ and x. Such x fulfills $\alpha_{\mathcal{G}}(x) = \gamma(x)$, for all $\gamma \in H$. Hence, $x \in \mathrm{A}_{\mathcal{G}}(H)$ as desired.

For (d): By (a), $(\hat{\mathcal{G}}/\mathrm{A}_{\mathcal{G}^\wedge}(H))^\wedge$ is topological group isomorphic to $\mathrm{A}_{\mathcal{G}^{\wedge\wedge}}(\mathrm{A}_{\mathcal{G}^\wedge}(H))$. Furthermore, it follows from (c) that $\mathrm{A}_{\mathcal{G}^{\wedge\wedge}}(\mathrm{A}_{\mathcal{G}^\wedge}(H))$ can be identified with $\mathrm{A}_{\mathcal{G}}(\mathrm{A}_{\mathcal{G}^\wedge}(H))$. Finally, from (b), we know that $\mathrm{A}_{\mathcal{G}}(\mathrm{A}_{\mathcal{G}^\wedge}(H)) = H$. Hence $(\hat{\mathcal{G}}/\mathrm{A}_{\mathcal{G}^\wedge}(H))^\wedge$ is topological group isomorphic to H, as desired.

To express in a more convenient way, the statement (a) in above theorem says that characters on $\mathcal{G}/H$ can be seen as the characters on $\mathcal{G}$, which which are equal to $1_{\mathbb{T}}$ on H. This property force the considered characters to be H-periodic.

3.8. Haar measure and Haar integral on LCA Group

We begin by defining a specific concept of measure on a topological group, which constitute the cornerstone for the Fourier analysis on abstract groups. But first we recall the notion of radon measure: Let $(\mathcal{X}, \mathfrak{B})$ be a Borel measureable space, and given there a measure μ. μ is said to be Borel measure , if to each $x \in \mathcal{X}$, there exists a measureable open neighborhood U of x, s.t. $\mu(U) < \infty$. From this definition it follows immediately that compact sets are of finite measures. A measure μ is called inner regular, if any Borel set B can be approximated from inside in measure arbitrarily well by means of compact subsets, i.e. $\forall B \in \mathfrak{B}$, $\forall \epsilon$, $\exists K \in \mathfrak{B}$, $K \subseteq B$ compact, s.t. $\mu(B\backslash K) < \epsilon$. The measure μ is called outer regular if any Borel set B can be approximated from outside in measure arbitrarily well by means of open sets, i.e. $\forall B \in \mathfrak{B}$, $\forall \epsilon > 0$, $\exists U \in \mathfrak{B}$, $B \subseteq U$ open, s.t. $\mu(U\backslash B) < \epsilon$. A Borel measure is called a Radon measure if it is inner - and outer regular.

Now we are ready to define the Haar measure in an appropriate way:

Definition 3.7 (Haar measure). *Let* $\mathcal{G}$ *be a topological group, and* $(\mathcal{G}, \mathfrak{B})$ *be a Borel measureable space. A Haar measure* μ *on* $\mathcal{G}$ *is defined as a non-zero Radon measure on* $\mathcal{G}$*, which is translation invariant, i.e.* $\mu(gA) = \mu(A)$*,* $\forall g \in \mathcal{G}$*, and* $A \in \mathfrak{B}$*.*

As we shall see soon, the translation-invariance property Haar measures is indispensable for our approach. By the following Theorem due to A. Haar, J. v. Neumann, and A. Weil, one can indeed be sure that such a measure in a Locally compact groups exists, and hence also in LCA group, and that such a measure is given uniquely up to a positive constant:

Theorem 3.37. *Let $\mathcal{G}$ be an LCA group. Then there is a (left) Haar measure on $\mathcal{G}$, which is unique up to a multiplication by a positive constant.*

Although the proof of above Theorem is not constructive, one can explicitly give in most cases a corresponding Haar measure. Some examples of Haar measures on LCA groups are given in the following:

Examples 3.38.

- *On $\mathbb{R}^N$, the Borel-Lebesgue measure is addition invariant, hence a Haar measure.*
- *On any countable LCA group, the counting measure is a Haar measure.*
- *Consider the quotient group $\mathbb{R}/\mathbb{Z}$. On this group, we can construct a Haar measure $\mu_{\mathbb{R}/\mathbb{Z}}$ induced from the Borel-Lebesgue measure $\mu_{\mathbb{R}}$ on $\mathbb{R}$ as follows: Let $A := [0, 1)$ be a measureable representation of $\mathbb{R}/\mathbb{Z}$ (we shall later call this set fundamental domain), i.e. each coset in $\mathbb{R}/\mathbb{Z}$ can be written uniquely as $[x]$, where $x \in A$ and let μ_A be a Borel measure on A induced from the Haar measure on $\mathbb{R}$. So we can define for $B \subseteq \mathbb{R}/\mathbb{Z}$, $\mu_{\mathbb{R}/\mathbb{Z}}(B) = \mu_{\mathbb{R}}(B \cap A)$. One can easily check that this measure is indeed a Haar measure.*

In some cases, to emphasize the fact that μ is a Haar measure on a LCA group $\mathcal{G}$, it is convenient to write μ_G instead of μ.

The following property of a Haar measure can be easily proven:

Proposition 3.39. *Let $\mathcal{G}$ be a LCA-group, and μ a Haar measure on $\mathcal{G}$. Then it holds:*

- *$\mathcal{G}$ is discrete if and only if $\mu_G(\{e\}) > 0$*
- *$\mathcal{G}$ is compact if and only if $\mu(\mathcal{G}) < \infty$*

If $\mathcal{G}$ is compact, it is customary from Thm. 3.37 and above proposition to consider the unique Haar measure normalized s.t. $\mu(\mathcal{G}) = 1$. We shall refer this as normalized Haar measure on $\mathcal{G}$. Let $\mathcal{G}$ be discrete, and fix a Haar measure $\mu_{\mathcal{G}}$. It follows from above proposition that the identity possessess a positive measure, and from translation-invariance of Haar measure, it follows that each element of $\mathcal{G}$ possessess the same measure as the identity. It is convenient to consider the Haar measure on $\mathcal{G}$, for which each element of $\mathcal{G}$ is assigned the measure 1, or equivalently for which $\mu_{\mathcal{G}}(\{e\}) = 1$. We refer this convention as counting measure on $\mathcal{G}$.

On account of the translation invariance of Haar measure, the orthogonality of the characters on compact LCA groups w.r.t. the inner product induced by the Haar measure can be easily established:

Lemma 3.40. *Let $\mathcal{G}$ be a compact LCA group. Then the elements of the dual group $\hat{\mathcal{G}}$ fulfill the following relation:*

$$\int\limits_{\mathcal{G}} (x, \gamma)\overline{(x, \gamma')}\mathrm{d}\mu_{\mathcal{G}}(x) = \mu_{\mathcal{G}}(\mathcal{G})\delta_{\gamma,\gamma'}, \tag{3.14}$$

for all $\gamma, \gamma' \in \hat{\mathcal{G}}$.

Proof. Since $\hat{\mathcal{G}}$ is a group, it is sufficient to prove above equality for $\gamma \in \hat{\mathcal{G}}$ and $\gamma' = 0$. The desired statement can be obtained by corresponding translation. For γ is also 0, the equality follows immediately. Let $\gamma \neq 0$. Clearly, there exist some $x_0 \in \mathcal{G}$ s.t. $(x_0, \gamma) \neq 1$. By computations:

$$\int\limits_{\mathcal{G}} (x, \gamma)\mathrm{d}\mu_{\mathcal{G}}(x) = (x_0, \mathcal{G}) \int\limits_{\mathcal{G}} (x - x_0, \gamma)\mathrm{d}\mu_{\mathcal{G}}(x) = (x_0, \gamma) \int\limits_{\mathcal{G}} (x, \gamma)\mathrm{d}\mu_{\mathcal{G}}(x),$$

where the second inequality follows from translation-invariance of Haar measure. So from above computation, it yields that $\int\limits_{\mathcal{G}} (x, \gamma)\mathrm{d}\mu_{\mathcal{G}}(x) = 0$. ∎

To ensures that the characters on a compact group constitute an orthonormal set, it seems likely to choose the Haar measure $\mu_{\mathcal{G}}$, s.t. $\mu_{\mathcal{G}}(\mathcal{G}) = 1$. We shall see soon, by means of the Fourier transform , that the characters form even an orthonormal basis for compact LCA group.

The following formula, due to Weil, gives rise about the connection between Haar measure of an LCA group $\mathcal{G}$, and the Haar measure of the quotient group $\mathcal{G}/K$, where K is a closed subgroup of $\mathcal{G}$:

Theorem 3.41 (Weil's Formula). *Let $\mathcal{G}$ be an LCA group, $K \leqslant \mathcal{G}$ be a closed subgroup. For $f \in L^1(\mathcal{G})$, the following holds:*

(a) For a.e. $x \in \mathcal{G}$, the function $k \mapsto f(xk)$ is μ_K-measureable and belongs to $L^1(K)$. The function $x \mapsto \int_K f(xk)\mathrm{d}\mu_K(k)$ depends only on the coset $[x] = xK$, i.e. it is constant on $[x]$. So it can be considered as a function $F : \mathcal{G}/K \to \mathbb{C}$, $[x] \mapsto \int_K f(xk)\mathrm{d}\mu_K(k)$.

(b) F is $\mu_{\mathcal{G}/K}$-measureable and $F \in L^1(\mathcal{G}/K)$. Furthermore, the Haar measures $\mu_{\mathcal{G}}$, μ_K, and $\mu_{\mathcal{G}/K}$ can be chosen s.t.:

$$\int_{\mathcal{G}} f(x)\mathrm{d}\mu_{\mathcal{G}}(x) = \int_{\mathcal{G}/K}\int_K f(xk)\mathrm{d}\mu_K(k)\mathrm{d}\mu_{\mathcal{G}/K}([x]). \tag{3.15}$$

By means of the Haar measure, which constitute an abstraction of Borel-Lebesgue measure in $\mathbb{R}^N$, it is now possible to define the Fourier transform of suitable functions on LCA-groups, as shall be done in the next section.

3.9. Fourier Transform

Definition 3.8 (Fourier Transform). *Let $\mathcal{G}$ be a LCA group, and let be $f \in L^1(\mathcal{G})$. The Fourier transform of f is defined as the map $\hat{f}$:*

$$\hat{f}(\gamma) = \int_{\mathcal{G}} f(x)\overline{\gamma(x)}\mathrm{d}\mu_{\mathcal{G}}(x), \quad \gamma \in \Gamma,$$

mapping from $\mathcal{G}$ to $\mathbb{C}$.

It is obvious, that the requirement $f \in L^1(\mathcal{G})$ is sufficient for the integrability of $f\gamma^*$ over whole $\mathcal{G}$, that the following inequality holds: $|\hat{f}(\gamma)| \leqslant \int_{\mathcal{G}}|f(x)|\mathrm{d}\mu_{\mathcal{G}}(x)$, $\forall\gamma \in \mathcal{G}^\wedge$ and hence the finiteness of $\hat{f}$ on almost every points of $\mathcal{G}^\wedge$. Furthermore, one can write the Fourier transform by means of a mapping $\mathcal{F} : L^1(\mathcal{G}) \to \mathcal{F}L^1(\mathcal{G})$, $f \mapsto \mathcal{F}f = \hat{f}$, whose linearity is obvious. As we have seen, the Fourier transform of an L^1-function is bounded, so it is convenient to equip $\mathcal{F}L^1(\mathcal{G})$ with the supremum norm. By this setting, the following norm inequality holds:

$$\|\hat{f}\|_{L^\infty(\mathcal{G}^\wedge)} \leqslant \|f\|_{L^1(\mathcal{G})}, \tag{3.16}$$

which shows that the Fourier transform is a norm-decreasing mapping from $L^1(\mathcal{G})$ to $\mathcal{F}L^1(\mathcal{G})$.

Involving an appropriate element of the space $C_c(\mathcal{G})$ (notice that this space is dense in $L^1(\mathcal{G})$), one can easily show that for each $f \in L^1(\mathcal{G})$, that $\hat{f}$ is (uniformly) continuous on $\mathcal{G}^\wedge$, in the sense that for each $\epsilon > 0$, there exists a compact subset $K \subseteq \mathcal{G}$ and an open neighborhood U of $1_{\mathbb{T}}$ (which is a "ball" of positive radius around $1_{\mathbb{T}}$ w.r.t. the modulus), s.t. for all $\gamma_1^{-1}\gamma_2 \in \mathcal{W}(K, U)$ (which is the neighborhood of "zero" $1_{\mathcal{G}^\wedge}$), it holds: $|\hat{f}(\gamma_1) - \hat{f}(\gamma_2)| < \epsilon$. To see this, let $f \in L^1(\mathcal{G})$, with f non-zero almost everywhere (in case that $f \equiv 0$ a.e. on $\mathcal{G}$, it clearly follows $\hat{f} \equiv 0$ a.e. on $\mathcal{G}^\wedge$), and let $\epsilon > 0$ be given. Now, take an $g \in C_c(\mathcal{G})$, with $\|g\|_{L^1} \neq 0$ s.t. $\|f - g\| < \epsilon/3$. Consider the neighborhood $\mathcal{W}'(1_{\mathcal{G}^\wedge}, K, \epsilon/\|g\|_{L^1})$ of $1_{\mathcal{G}^\wedge}$, where $K = \mathrm{supp}(g)$. For $\gamma_1, \gamma_2 \in \mathcal{G}^\wedge$ with $\gamma_1^{-1}\gamma_2$, one can compute:

$$|\hat{f}(\gamma_1) - \hat{f}(\gamma_2)| \leqslant |\hat{f}(\gamma_1) - \hat{g}(\gamma_1)| + |\hat{g}(\gamma_1) - \hat{g}(\gamma_2)| + |\hat{f}(\gamma_2) - \hat{g}(\gamma_2)|. \tag{3.17}$$

The first - and the third summand are, by the choice of g and by the fact that characters are of modulus 1, is less than $\epsilon/3$. For the second summand, one can compute:

$$\begin{aligned}|\hat{g}(\gamma_1) - \hat{g}(\gamma_2)| &\leqslant \int_{\mathcal{G}} |g(x)||\gamma_2^{-1}(x)||\gamma_1^{-1}\gamma_2(x) - 1| \mathrm{d}\mu_{\mathcal{G}}(x) < \int_{K} |g(x)||\gamma_1^{-1}\gamma_2(x) - 1| \mathrm{d}\mu_{\mathcal{G}}(x) \\ &< \|g\|_{L^1} \frac{\epsilon}{\|g\|_{L^1}} = \frac{\epsilon}{3}.\end{aligned}$$

So, the desired statement holds.

Furthermore, one can show that the Fourier transform of an absolut-integrable function f on $\mathcal{G}$ vanish at infinity[3], in the following sense:

Definition 3.9. *Let $\mathcal{X}$ be a Hausdorff topological space, and $f : \mathcal{X} \to \mathbb{C}$ be a function. f is said to vanish at infinity if for all $\epsilon > 0$, there exists $K = K(\epsilon) \subseteq \mathcal{X}$ compact, s.t. $|f(x)| < \epsilon$, for every point $x \in \mathcal{X}\backslash K$. The set of all functions on $\mathcal{X}$ vanishing at infinity is denoted by $C_0(X)$. Furthermore, it is convenient to give $C_0(X)$ the supremum norm $\|f\|_\infty := \sup_{x\in\mathcal{X}}|f(x)|$, which makes $C_0(\mathcal{X})$ complete.*

Since $\hat{f}$ is continuous on $\mathcal{G}^\wedge$ and the image of compact sets under continuous function is clearly compact, to show that $\hat{f}$ vanishes at infinity, it is sufficient to show that for each $\epsilon > 0$, the set $\{\gamma \in \mathcal{G}^\wedge : |\hat{f}(\gamma)| \geqslant \epsilon\}$ is compact in $\mathcal{G}^\wedge$. One might making use of continuity of the translation map (which is not hard to establish), and Ascoli's Theorem to show this fact.

So it is now convenient to see the Fourier transform as the mapping $\mathcal{F} : L^1(\mathcal{G}) \to C_0(\mathcal{G})$. Furthermore, it is immediate to see that $\mathcal{F}$ is linear and bounded. Hence one can see the Fourier transform as a bounded operator between $L^1(\mathcal{G})$ into $C_0(\mathcal{G})$.

By some computations involving Fubini Theorem and the translation-invariance of the Haar measure, one obtains immediately the following equality:

$$\mathcal{F}(f * g) = \widehat{f * g} = f \cdot g, \quad \text{for } f, g \in L^1(\mathcal{G}). \tag{3.18}$$

Consider the set $\mathcal{F}(L^1(\mathcal{G}))$ of all Fourier transform of functions in $L^1(\mathcal{G})$. By means of Stone-Weierstrass theorem, we shall in the following show that $\mathcal{F}(L^1(\mathcal{G}))$ is dense in $C_0(\mathcal{G}^\wedge)$, hence $\mathcal{F}$ can be continuously extendeded to a bounded surjective operator between $L^1(\mathcal{G})$ and $C_0(\mathcal{G})$. In particular, we need to show that $\mathcal{F}(L^1(\mathcal{G}))$ is closed under involution (which is in this case the usual complex conjugation inherited from $C_0(\mathcal{G}^\wedge)$) and a separating subalgebra of $C_0(\mathcal{G}^\wedge)$ (equipped with usual pointwise algebraic operations and the supremum norm). It is not hard to see that $\mathcal{F}(L^1(\mathcal{G}))$ is a subalgebra of $C_0(\mathcal{G}^\wedge)$ (This follows from 3.18 and the fact that $L^1(\mathcal{G})$ equipped with the convolution is an algebra). The fact that $\mathcal{F}(L^1(\mathcal{G}))$ is closed under complex conjugation follows from the fact that $L^1(\mathcal{G})$ is closed under the involution $f(\cdot)^* = \overline{f(-(\cdot))}$. That $\mathcal{F}(L^1(\mathcal{G}))$ separates points of $\mathcal{G}^\wedge$ is easy to see. Indeed, since $\mathcal{F}$ is linear, one has only to ensure that $\forall\gamma \in \mathcal{G}^\wedge$ not equal the characteristic function on $\mathbb{T}$, there exist an $f \in L^1(\mathcal{G})$ s.t. $\hat{f}(\gamma) \neq 0$. The fact, that the Fourier transform is an surjective operator between $L^1(\mathcal{G})$ and $C_0(\mathcal{G}^\wedge)$ can be seen as an abstract form of Riemann-Lebesgue-Lemma.

From the translation invariance of the Haar measure, it follows immediately:

$$\text{For } f \in L^1(\mathcal{G}), \text{ and } x_0 \in \mathcal{G} : \quad f(\widehat{(\cdot) - x_0})(\gamma) = \overline{(x_0, \gamma)}\hat{f}(\gamma), \quad \gamma \in \hat{\mathcal{G}}, \tag{3.19}$$

[3]This formal definition of functions vanishing at infinity is motivated by the following observation: Consider the Alexandrov compactification of the Hausdorff space $\mathcal{X}$, i.e. $\mathcal{X}^\infty := \mathcal{X} \cup \{\infty\}$, where ∞ denotes the point which is not contained in $\mathcal{X}$, e.g. the infinity. It is natural to give $\mathcal{X}^\infty$ the topology, whose open set U is either an open set in $\mathcal{X}$, or $\infty \in U$ and $\mathcal{X}\backslash U$ is compact in $\mathcal{X}$. Recall that compact subsets of $\mathcal{X}$ is closed. Continuous functions on $\mathcal{X}$ can be seen as the restriction of continuous functions on $\mathcal{X}^\infty$. By this reason, one can identify $C_0(\mathcal{X})$ with the space of all continuous functions f on $\mathcal{X}$, for which $f(\infty) = 0$.

and from the basic properties of characters:

$$\text{For } f \in L^1(\mathcal{G}), \text{ and } \gamma_0 \in \hat{\mathcal{G}}: \quad \widehat{(x,\gamma_0)f}(x) = \hat{f}((\cdot) - \gamma_0). \tag{3.20}$$

The equality in (3.20) is simply an abstract form of the signal modulation in $\mathbb{R}^N$.

Now, we summarized the discussions done previously in the following theorem:

Theorem 3.42. *Let $\mathcal{G}$ be an LCA group. Then the following holds:*

(a) $\mathcal{F} : L^1(\mathcal{G}) \to C_0(\mathcal{G})$ is a bounded surjective operator.

(b) For $f \in L^1(\mathcal{G})$, $x_0 \in \mathcal{G}$ and $\gamma_0 \in \hat{\mathcal{G}}$, it holds $\widehat{f((\cdot) - x_0)}(\gamma) = \overline{(x_0, \gamma)}\hat{f}(\gamma)$, for $\gamma \in \hat{\mathcal{G}}$, and $\widehat{(x,\gamma_0)f}(x) = \hat{f}((\cdot) - \gamma_0)$.

We shall see soon that Fourier transform is a unitary equivalence between $L^2(\mathcal{G})$ and $L^2(\mathcal{G}^\wedge)$.

There exists a connection between Fourier transform and positive-definite functions. In the following, we give for convenient the notion of positive-definite functions on LCA groups:

Definition 3.10. *Let $\mathcal{G}$ be an LCA group written additively, and $\phi : \mathcal{G} \to \mathbb{C}$. ϕ is said to be positive-definite if:*

$$\sum_{m,n \in [N]} c_n \overline{c_m} \phi_{x_n - x_m} \geqslant 0,$$

where $N \in \mathbb{N}$, $c_k \in \mathbb{C}$, and $x_k \in \mathcal{G}$, for all $k \in [N]$.

In particular, one have a relatively good guess of the shape of such class of functions:

Lemma 3.43. *Let $\phi : \mathcal{G} \to \mathbb{C}$ be a positive definite function on an LCA group written additively. Then the following holds:*

(a) ϕ is hermitian on $\mathcal{G}$, i.e. $\phi(-x) = \overline{\phi(x)}$, $\forall x \in \mathcal{G}$.

(b) $|\phi(x)| \leqslant \phi(0_{\mathcal{G}})$, $\forall x \in \mathcal{G}$, in particular $\phi(0_{\mathcal{G}}) \geqslant 0$ and ϕ is a bounded function

(c) $|\phi(x) - \phi(y)|^2 \leqslant 2\phi(0)\Re(\phi(0_{\mathcal{G}}) - \phi(x - y))$, $\forall x, y \in \mathcal{G}$.

The corresponding connection between such type of functions and the Fourier transform is given by the following theorem:

Theorem 3.44 (Bochner's Theorem). *Let $\mathcal{G}$ be an LCA group. Given a continuous function $\phi : \mathcal{G} \to \mathbb{C}$. Then the following statements are equivalent:*

(a) ϕ is positive-definite

(b) there exists a non-negative bounded regular complex valued measure μ on $\mathcal{G}$ s.t. ϕ can be represented by:

$$\phi(x) = \int_{\mathcal{G}^\wedge} \gamma(x) \mathrm{d}\mu(x), \quad x \in \mathcal{G}.$$

Given $h \in L^1(\mathcal{G}^\wedge)$. The inverse Fourier transform of h can be defined as:

$$\check{h}(x) := \int_{\hat{\mathcal{G}}} h(\gamma)(x,\gamma) \mathrm{d}\mu_{\hat{\mathcal{G}}}(\gamma). \tag{3.21}$$

The following characterization of inverse Fourier transform can be easily established:

Lemma 3.45. *The inverse Fourier transform $\mathcal{F}^{-1}$ is a bounded surjective operator between $L^1(\mathcal{G}^\wedge)$ and $C_0(\mathcal{G})$.*

Proof. Let $h \in L^1(\mathcal{G}^\wedge)$. Let $\tilde{\mathcal{F}}$ be the Fourier transformation mapping from functions on $\mathcal{G}^\wedge$ to functions on $\mathcal{G}^{\wedge\wedge}$. The image of h under $\tilde{\mathcal{F}}$ is correspondingly:

$$\tilde{h}(\xi) := (\tilde{\mathcal{F}}h)(\xi) = \int_{\mathcal{G}^\wedge} h(\gamma)\overline{\xi(\gamma)}\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma), \quad \xi \in \mathcal{G}^{\wedge\wedge}.$$

By Pontryagin-van Kampen duality, we can identify elements of $\mathcal{G}^{\wedge\wedge}$ with an element of $\mathcal{G}$ by the natural mapping $\alpha_{\mathcal{G}}$. Hence $\tilde{\xi}$ can be seen as a function on $\mathcal{G}$:

$$\tilde{h}(x) := \tilde{h}(\alpha_{\mathcal{G}}(x)) = \int_{\mathcal{G}^\wedge} h(\gamma)\overline{\alpha_{\mathcal{G}}(x)(\gamma)}\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma) = \int_{\mathcal{G}^\wedge} h(\gamma)\overline{\gamma_x}\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma),$$

for all $x \in \mathcal{G}$. So, $\tilde{\mathcal{F}}$ is basically $\mathrm{V} \circ \mathcal{F}^{-1}$, where V denotes the reflection mapping, i.e. $\mathrm{V}f = f(-(\cdot))$, for $f \in L^1(\mathcal{G})$. It can be easily seen that V is an isometric isomorphism. Hence $\tilde{\mathcal{F}}$ is a mapping from $L^1(\mathcal{G})$ to $C_0(\mathcal{G})$. ∎

By some efforts, one may be able to establish the following theorem:

Theorem 3.46 (Fourier Inversion Formula). *Let $\mathcal{G}$ be an LCA group, and let a Haar measure $\mu_{\mathcal{G}}$ be chosen. For $f \in L^1(\mathcal{G})$, s.t. $\hat{f} \in L^1(\mathcal{G})$, one can adjust the Haar measure $\mu_{\mathcal{G}^\wedge}$ on $\mathcal{G}^\wedge$ s.t. the following formula holds:*

$$f(x) = \int_{\mathcal{G}^\wedge} \hat{f}(\gamma)\gamma(x)\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma) = \hat{\hat{f}}(-x), \quad x \in \mathcal{G}.$$

In case that the inversion formula holds, the Fourier transform as an operator from $L^1(\mathcal{G}) \cap L^2(\mathcal{G})$ can be extended to a unitary operator mapping between $L^2(\mathcal{G})$ onto $L^2(\hat{\mathcal{G}})$, where $L^2(\hat{\mathcal{G}})$ denotes the Hilbert space containing equivalence classes (modulo $\mu_{\hat{\mathcal{G}}}$-null sets) of square-integrable complex functions on Γ. Furthermore, in this case the Parseval formula holds, i.e. $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle\ \forall f, g \in L^2(\mathcal{G})$, as a consequence of the mentioned unitarity of $\mathcal{F} : L^2(\mathcal{G}) \to L^2(\hat{\mathcal{G}})$:

Theorem 3.47 (Plancherel Theorem). *Let $\mathcal{G}$ be an LCA group, and $\mu_{\mathcal{G}}$ a Haar measure on $\mathcal{G}$. Then there exists a unique Haar measure on $\widehat{\mathcal{G}}$, s.t. for all $f \in L^1(\mathcal{G}) \cap L^2(\mathcal{G})$ it holds:*

$$\|f\|_{L^2(\mathcal{G})} = \|f\|_{L^2(\widehat{\mathcal{G}})}.$$

Furthermore, Fourier transform can be extended to a unitary operator mapping between $L^2(\mathcal{G})$ to $L^2(\widehat{\mathcal{G}})$.

For further approaches, we shall use the Fourier transform as unitary equivalence between $L^2(\mathcal{G})$ and $L^2(\mathcal{G}^\wedge)$. In particular, we need the following characterization of the Fourier transform on compact abelian groups:

Corollary 3.48. *Let $\mathcal{G}$ be a compact abelian groups s.t. its dual is countable, and choose the Haar measure on $\mathcal{G}$ as the normalized one. Then $\mathcal{G}^\wedge$ is an orthonormal basis for $L^2(\Omega)$*

The following proposition is of particular nature for the existence of orthonormal basis for functions in $L^2(\mathcal{G})$, where $\mathcal{G}$ is compact.

Proposition 3.49. *Let $\mathcal{G}$ be an LCA group. $\mathcal{G}$ is compact and metrizable (in particular, second countable) if and only if $\mathcal{G}^\wedge$ is countable.*

Proof. "$\Rightarrow$":Let $\mathcal{G}$ be compact and metrizable. Then it follows that $\mathcal{G}^\wedge$ is compactly generated. So write $\mathcal{G}^\wedge = \bigcup_{n\in\mathbb{N}} K_n$, where $K_n \subseteq \mathcal{G}^\wedge$ is compact, $\forall n \in \mathbb{N}$. Furthermore, $\mathcal{G}^\wedge$ is discrete since $\mathcal{G}$ is compact (thm. 3.6). Hence, each K_n, $n \in \mathbb{N}$, must be finite. Finally, noticing that countable union of finite sets is countable, the desired statement is shown.

"$\Leftarrow$": For converse, let $\mathcal{G}^\wedge$ be countable. By the Pontryagin-van Kampen duality theorem, $\mathcal{G}$ can be identified with $\mathcal{G}^{\wedge\wedge}$. Hence, $\mathcal{G}$ is compact, since $\mathcal{G}^{\wedge\wedge}$ is compact (this follows from countability, and in particular discreteness, of $\mathcal{G}^\wedge$) ∎

4. Orthonormal Bases of Exponentials and the Sampling Problem

An abstraction of the notion band-limited finite-energy signal shall be given in the following definition:

Definition 4.1 (Paley-Wiener Spaces). *Let $\mathcal{G}$ be an LCA group, $\Omega \subseteq \mathcal{G}^\wedge$ be a measureable subset, and $f : \mathcal{G} \to \mathbb{C}$ be a measureable function. Assume that f is square-integrable, i.e. $f \in L^2(\Omega)$. f is said to be band-limited to $\Omega \subseteq \mathcal{G}^\wedge$, if supp$(\hat{f}) \subseteq \Omega$. The set of such functions band-limited to Ω, equipped with the inner product structure inherited from $L^2(\mathcal{G})$, is called Paley-Wiener space, and is denoted by $\mathcal{PW}^2_\Omega$.*

One can immediately see that the space of band-limited function on $\mathbb{R}^N$ is just a special case of this definition. From the fact that $L^2(\Omega)$ as a closed subspace of $L^2(\mathcal{G}^\wedge)$, and by definition $\mathcal{F}^{-1}(L^2(\Omega)) = \mathcal{PW}^2_\Omega$, it follows immediately that Lemma 2.2 asserts that $\mathcal{F}^{-1} : L^2(\Omega) \to \mathcal{PW}^2_\Omega$ is again a unitary equivalence.

For the phase retrieval problem, which shall be treated later, it is convenient to consider a slight modification of Paley-Wiener spaces:

Definition 4.2. *Let $\mathcal{G}$ be an LCA group, and $\tilde{\Omega} \subseteq \mathcal{G}$ be a measureable subset. The Paley-Wiener space $\tilde{\mathcal{PW}}^2_\Omega$ consists of measureable and square-integrable functions f on $\mathcal{G}^\wedge$, whose inverse Fourier transform is supported in Ω.*

Again by the similar argumentation for the unitarity of $\mathcal{F}^{-1} : L^2(\Omega) \to \mathcal{PW}^2_\Omega$, it follows that $\mathcal{F} : L^2(\tilde{\Omega}) \to \tilde{\mathcal{PW}}^2_{\tilde{\Omega}}$ is a unitary operator.

In particular, we shall study the sampling behaviour of $\mathcal{PW}^2_\Omega$ by studying the natural orthonormal basis of $L^2(\Omega)$, which is formed by the collection of functions on $\Omega \subseteq \mathcal{G}^\wedge$ inherited from a quotient group $\mathcal{G}^\wedge/\Lambda$, where Λ is a (necessary for our approach: discrete countable) subgroup of $\mathcal{G}$. In particular, Ω and $\mathcal{G}^\wedge/\Lambda$ should possess the following relation: Ω is a relatively compact measureable subset of $\mathcal{G}^\wedge$ which ("almostly" in Lebesgue sense) contitute a representation of each coset of $\mathcal{G}^\wedge/H$. Once the orthonormal basis for $L^2(\Omega)$ is established, the sampling and reconstruction formula follows immediately from the orthonormal expansion of the Fourier transform of each function in $L^2(\Omega)$ and the inverse Fourier transform of this expression. This fact is justified by the bases -, and correlation preserving property of the inverse Fourier transform as a unitary equivalence between $L^2(\Omega)$ and $\mathcal{PW}^2_\Omega$. Furthermore, it shall be shown the approximation $\tilde{f}$ of a function $f \in L^2(\mathcal{G})$ resulted from this process is a continuous function on $\mathcal{G}$, and is equal to f in Lebesgue sense, i.e. $\tilde{f} = f$ a.e. on $\mathcal{G}$. From the practical point of view, this is clearly not a serious problem, since real-world signals (in particular, those which appears in the application of electrical engineering) are continuous. The approaches made in this chapter follows from [31].

In this chapter, we write, if not otherwise stated, all the LCA groups additively, and the identity of an LCA group $\mathcal{G}$ will be denoted by $0_\mathcal{G}$. For convenient, the dual group $\mathcal{G}^\wedge$ of an LCA group is written multiplicatively, and the identity is denoted by $1_{\mathcal{G}^\wedge}$. In the following section, we familiarize ourselves with the notion of uniform lattices in an LCA groups, which constitute a fundamental for our approach.

4.1. Lattices on LCA group

We first give the definition of the uniform lattice in the following definition:

Definition 4.3 ((Admissible) Uniform Lattice). *Let $\mathcal{G}$ be an LCA group. $\Lambda \subseteq \mathcal{G}$ is said to be a uniform lattice on $\mathcal{G}$, if Λ is a discrete subgroup of $\mathcal{G}$, and $\mathcal{G}/\Lambda$ is compact. We say, a uniform lattice is admissible*[1] *if it is countable, and its annihilator is also countable.*

For a uniform lattice Λ, one can immediately give a characterization of its annihilator:

Lemma 4.1. *Let $\mathcal{G}$ be an LCA group, and Λ a uniform lattice. Then $\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)$ is also a uniform lattice in $\mathcal{G}^\wedge$*

Proof. Clearly, $\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)$ is discrete. $(\mathcal{G}^\wedge/\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda))^\wedge$ is discrete, since this LCA group is topological group isomorphic to Λ. Since $(\mathcal{G}^\wedge/\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda))^\wedge$ is discrete, it follows immediately that $(\mathcal{G}^\wedge/\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda))^{\wedge\wedge}$ is compact. Finally by Pontryagin-van Kampen duality, it follows that $\mathcal{G}^\wedge/\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)$ is compact, and accordingly the desired statement hold. ∎

Further, in case that the considered LCA group is in some sense "ordinary", there is no need to restrict the class of considered uniform lattice, since:

Lemma 4.2. *Let $\mathcal{G}$ be a second-countable LCA group, and Λ a uniform lattice in $\mathcal{G}$. Then the following statements hold:*

(a) Λ is countable.

(b) The annihilator $\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)$ of Λ in $\mathcal{G}^\wedge$ is countable.

Sketch of Proof. (a):As a discrete subset of a second-countable LCA group, Λ is automatically countable.

(b):We already know that $\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)$ is discrete. Furthermore, the second countability assumption on $\mathcal{G}$ implies that $\mathcal{G}^\wedge$ is second-countable. Hence, as a subset of $\mathcal{G}^\wedge$, $\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)$ is countable. From the fact that $\mathcal{G}^\wedge$ is second-countable ,since $\mathcal{G}$ is countable, it follows immediately that $\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)$ is countable. ∎

The following notions are indispensable for our later aproaches:

Definition 4.4. *Let $\mathcal{G}$ be an LCA group, and $K \subseteq \mathcal{G}$ a closed subgroup. A (Borel) measureable set $\Omega \subseteq \mathcal{G}$ is said to be a (measureable) transversal of $\mathcal{G}/K$, if Ω is the set of representatives (transversal) of $\mathcal{G}/K$, i.e. $\#((x+K) \cap \Omega) = 1$. To each (measureable) transversal, there corresponds a mapping $\varsigma : \mathcal{G}/K \to \Omega$, $[x] \mapsto (x+H) \cap \Omega$, which sends each coset to the corresponding representative in the fundamental domain. Furthermore, ς is called the cross-section map.*

For convenient, we call measureable transversal simply transversal. For easeness, let Ω be a transversal of the quotient group $\mathcal{G}/K$, then we say also Ω is a fundamental domain of K. The following obvious statements are alternative characterizations of transversals:

Proposition 4.3. *Let $\mathcal{G}$ be an LCA group, and $K \leqslant \mathcal{G}$ a closed subgroup. Given a measureable subset $\Omega \subseteq \mathcal{G}$. Then the following statements are equivalent*

(a) Ω is a measureable transversal of the quotient group of K.

(b) The canonical quotient mapping $\mathrm{q} : \mathcal{G} \to \mathcal{G}/K$ restricted to Ω, i.e. $\mathrm{q}|_{\Omega}$, is bijective.

(c) $\mathcal{G} = \Omega + K$ and $\Omega \cap (\Omega + k) = \varnothing$, $\forall k \in K \backslash \{0_K\}$.

[1] In the sense, that the uniform lattice is admissible for our purpose, to establish an orthonormal basis for the Lebesgue space $L^2(\mathcal{G}/\Lambda)$ on the quotient group $\mathcal{G}/\Lambda$, where $\mathcal{G}$ is an LCA group, and Λ a uniform lattice

The existence of a "good-natured" fundamental domain is ensured by the following lemma:

Lemma 4.4. *Let $\mathcal{G}$ be an LCA group and Λ is a uniform lattice. Then there exists a relatively compact transversal for $\mathcal{G}/\Lambda$.*

The existence of a measureable representation of $\mathcal{G}/\Lambda$ was first ensured in Thm. 1 in [12], and the existence of a relatively compact transversal in Lemma 2 in [30].

Since integrals is involved in our approach, and integrals can not "see" measure zero sets, it seems likely to slacken/generalize the definition of fundamental domain in the following way:

Definition 4.5. *Let $\mathcal{G}$ be an LCA group, and $\Lambda \leqslant \mathcal{G}$ be a uniform lattice. A measureable subset $\Omega \subseteq \mathcal{G}$ is said to be an almost transversal of $\mathcal{G}/\Lambda$, if Ω differs to a fundamental domain of Λ in a $\mu_{\mathcal{G}}$-null set. We say also Ω is an almost fundamental domain of Λ.*

Equivalently, one can define an almost fundamental domain Ω of a uniform lattice Λ in an LCA group as follows: almost every $x \in \mathcal{G}$ can be written uniquely as $x = d + \lambda$, where $d \in \Omega$ and $\lambda \in \Lambda$. This equivalent definition is justified by the definition of almost fundamental domain, translation invariance of the Haar measure, closeness of measureble subset under countable intersection and union, and the countable additivity of measures. As we seen previously, the notion of almost measureable transversal is closely related with the notion of the so-called K-tiling subset of $\mathcal{G}$. We define this term as follows:

Definition 4.6. *Let $\mathcal{G}$ be an LCA group and $K \in \mathbb{N}$. Given a measureable subset $\Omega \subseteq \mathcal{G}$ and a countable subset $\Lambda \subset \mathcal{G}$. Ω is said to K-tile $\mathcal{G}$ by (translation set) Λ if:*

$$\sum_{\lambda \in \Lambda} \chi_{\Omega}(x - \lambda) = K, \quad \text{a.e. } x \in \mathcal{G}.$$

If above condition is fulfilled, then Ω is said to be K-tiling subset of $\mathbb{R}^N$ by Λ. In case that $K = 1$, one can simply say: Ω tiles $\mathbb{R}^N$ with Λ, and write: $\Omega + \Lambda = \mathcal{G}$.

By means of above notion, one can translate an algebraic notion (Fundamental domain) to a more "descriptive" notion (Tiling set).

4.2. Lattice and Measure equivalent sets - Fundamental Domains

In the following section we concern ourselves with two notions of equivalents between fundamental domains of an admissible uniform lattice. Firstly, the notion of partition equivalent of fundamental domains of a lattice, and secondly, the notion of measure equivalent of fundamental domains of a lattice.

4.2.1. Fundamental domains - lattice equivalent sets

We first give in the following, the notion of lattice equivalent sets:

Definition 4.7 (Λ-equivalent set). *Let $\mathcal{G}$ be an LCA group, and let $\Lambda \leqslant \mathcal{G}$ be a discrete countable subgroup. Given two subsets A and B of $\mathcal{G}$. A and B is said to be Λ-equivalent, if there exists two measureable countable partitions $\{A_n\}_{n\in\mathcal{I}}$ and $\{B_n\}_{n\in\mathcal{I}}$ of A and B respectively, and a labelling $\{\lambda_n\}_{n\in\mathcal{I}}$ of the elements of Λ by $\mathcal{I}$, s.t. $A_n + \lambda_n = B_n$, $\forall n \in \mathcal{I}$. We write $A \sim_\Lambda B$.*

One can show that fundamental domains of a lattice can be seen as equivalent in the following sense:

Lemma 4.5. *Let $\mathcal{G}$ be an LCA group, and Λ be a discrete countable subgroup of $\mathcal{G}$. Given a transversal Ω of $\mathcal{G}/\Lambda$. Then a subset $\tilde{\Omega} \subseteq \mathcal{G}$ is also a transversal of $\mathcal{G}/\Lambda$, if and only if $\tilde{\Omega} \sim_\Lambda \Omega$.*

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a labelling of Λ.

"$\Rightarrow$" We write $\tilde{\Omega}_n := (\Omega + \lambda_n) \cap \tilde{\Omega}$, $\forall n \in \mathbb{N}$. Notice that $\{\tilde{\Omega}_n\}_{n\in\mathbb{N}}$ forms a measureable partition of $\tilde{\Omega}$, since Ω is a transversal of $\mathcal{G}/\Lambda$. In return, define $\Omega_n = (\tilde{\Omega} - \lambda_n) \cap \Omega$, $\forall n \in \mathbb{N}$. Notice that again since $\tilde{\Omega}$ is a transversal of $\mathcal{G}/\Lambda$, it follows immediately that $\{\Omega_n\}_{n\in\mathbb{N}}$ forms a measureable partition of Ω. By the definition of both partitions, it follows immediately that $\Omega_n + \lambda_n = \tilde{\Omega}_n$, $\forall n \in \mathbb{N}$. Hence the statement holds.

"$\Leftarrow$" Now, suppose that there exists a measureable partitions $\{\Omega_n\}_{n\in\mathbb{N}}$ of Ω, and $\{\tilde{\Omega}_n\}_{n\in\mathbb{N}}$ of $\tilde{\Omega}$, s.t. $\Omega_n + \lambda_n = \tilde{\Omega}_n$. Let $x \in \mathcal{G}$ be arbitrary. Since Ω is a transversal of $\mathcal{G}/\Lambda$, it follows that there exists a unique representation of x by an $\omega \in \Omega$ and λ_m, for exactly a $m \in \mathbb{N}$, s.t. $x = \omega + \lambda_m$. Since $\{\Omega_n\}_n$ forms a partition of Ω, it follows that there exists exactly a $n \in \mathbb{N}$, s.t. $\omega \in \Omega_n$. From the Λ-equivalence of Ω and $\tilde{\Omega}$, one can imply that x can be written as $x = \tilde{\omega} + \lambda_{\tilde{m}}$, where $\tilde{\omega} := \omega + \lambda_n \in \tilde{\Omega}_n$ unique, and $\lambda_{\tilde{m}} := \lambda_m - \lambda_n$. Now suppose that there exists another $\tilde{m}' \in \mathbb{N}$ s.t. $x = \tilde{\omega} + \lambda_{\tilde{m}'}$ holds. We can write $x = \omega + \lambda_n + \lambda_{\tilde{m}'}$. So, it must hold $\lambda_n + \lambda_{\tilde{m}'} = \lambda_m$, and hence $\lambda_{\tilde{m}'} = \lambda_m - \lambda_n = \lambda_{\tilde{m}}$. So each $x \in \mathcal{G}$ has a unique representation by means $\tilde{\Omega}$ and Λ, and correspondingly $\tilde{\Omega}$ is a transversal for $\mathcal{G}/\Lambda$. ∎

Since Fundamental domains of a lattice can be seen as "partitions" equivalent, it stands clear that the following mapping is defined canonically:

Definition 4.8. *Let $\mathcal{G}$ be an LCA group, and Λ a discrete countable subgroup of $\mathcal{G}$. Let two fundamental domains Ω, and Ω' be given. The mapping $\tau_{\Omega\to\Omega'} : \Omega \to \Omega'$, $x \mapsto (x + \Lambda) \cap \Omega'$, is called cross-transversal mapping.*

Since Ω and Ω' are fundamental domains of Λ, it follows immediately that $(x + \Lambda) \cap \Omega'$ contains only a singleton $\{x + \lambda\}$, for a $\lambda \in \Lambda$. Furthermore for such an $x \in \Omega$ and the corresponding $\lambda \in \Lambda$, we set $\tau_{\Omega\to\Omega'}(x) = x + \lambda$. Hence $\tau_{\Omega\to\Omega'}$ is appropriately defined. Furthermore, since subgroups are closed under taking inverse, $\tau_{\Omega\to\Omega'}$ can also alternatively be written as $\tau_{\Omega\to\Omega'}(x) = (x - \lambda) \cap \Omega'$. It is obvious that $\tau_{\Omega\to\Omega'}$ is bijective, and that its inverse is given by $\tau^{-1}_{\Omega\to\Omega'} = \tau_{\Omega'\to\Omega}$.

By lemma 4.5, $\tau_{\Omega\to\Omega'}$ can be specified as follows: Since $\Omega \sim_\Lambda \Omega'$, there exists a countable measureable partition $\{\Omega_n\}_{n\in\mathcal{I}}$ of Ω, a countable measureable partition $\{\Omega'_n\}_{n\in\mathcal{I}}$ of Ω', and an indexing $\{\lambda_n\}_{n\in\mathcal{I}}$ by Λ, s.t. $\Omega_n + \lambda_n = \Omega'_n$, for all $n \in \mathcal{I}$. So for the mapping $\tau_{\Omega\to\Omega'}$, it holds: $\tau_{\Omega\to\Omega'}(x) = x + \lambda_n \in \Omega'_n$, for $x \in \Omega_n$, $n \in \mathcal{I}$. Clearly, its inverse is correspondingly $\tau_{\Omega\to\Omega'}(x') = x' - \lambda_n \in \Omega'_n$, for $x' \in \Omega'$.

By the following easy statement and corresponding geometric restriction of the considered fundamental domains, one can "specify" lemma 4.5 as follows:

Lemma 4.6. *Let $\mathcal{G}$ be an LCA group, K, $\tilde{K}$ be compact subsets of $\mathcal{G}$, and Λ be a countable subgroup of $\mathcal{G}$. Define the set:*

$$\Lambda' := \{\lambda \in \Lambda : (\lambda + K) \cap \tilde{K} = \varnothing\}.$$

Then Λ' is finite.

Proof. notice that $\Lambda' \subseteq \Lambda \cap (\tilde{K} - K)$. From the fact that finite sum of compact sets is compact, and intersection of compact subset with countable set yields a finite set, Λ' is finite. ∎

As desired, we obtain the following statement, which is an immediate consequence of the proof of lemma 4.5, and above statement:

Lemma 4.7. *Let $\mathcal{G}$ be an LCA group, and Λ be a discrete countable subgroup of $\mathcal{G}$. Given a relatively compact transversal Ω of $\mathcal{G}/\Lambda$. Then a subset $\tilde{\Omega} \subseteq \mathcal{G}$ is also a relatively compact transversal of $\mathcal{G}/\Lambda$, if and only if there exists a finite subset $\tilde{\Lambda}$ of Λ, s.t. $\Omega \sim_{\tilde{\Lambda}} \tilde{\Omega}$.*

Let Ω and Ω' be relatively compact fundamental domains of a discrete subgroup Λ of Ω. Let $\{\Omega_n\}_{n\in[M]}$ and $\{\Omega'_n\}_{n\in[M]}$ be the finite measureable partitions of Ω and Ω', and $\{\lambda_n\}_{n\in[M]}$ be the finite subset of Λ, which are asserted by lemma 4.7. Obviously, the cross-transversal mapping can in this case be written as $\tau_{\Omega \to \Omega'}(x) = x + \lambda_n \in \Omega_n^{'}$, for $x \in \Omega_n$, where $n \in [M]$.

In the next subsection, we shall show that all relatively compact fundamental domains of a certain countable lattice possesses the same measure. In particular, it will be shown, that by this observation, one can give a canonical unitary operator, mapping between signal spaces, each defined on one of those fundamental domains.

4.2.2. Fundamental domains - measure equivalent sets

Recall that cross-section map $\varsigma : \mathcal{G}/\Lambda \to \Omega$ corresponding to a uniform lattice Λ in an LCA group $\mathcal{G}$ and a relatively compact fundamental domain Ω of Λ is defined as the mapping $[x] \mapsto (x + \Lambda) \cap \Omega$. The following lemma gives a useful property of ς:

Lemma 4.8. *Let $\mathcal{G}$ be an LCA group, Λ be a countable uniform lattice in $\mathcal{G}$, and Ω a relatively compact fundamental domain of Λ. Furthermore, fix the Haar measures μ_Λ s.t. μ_Λ is a counting measure, $\mu_{\mathcal{G}/\Lambda}$ s.t. $\mu_{\mathcal{G}/\Lambda}(\mathcal{G}/K) = 1$. By the previous choices of μ_Λ and $\mu_{\mathcal{G}/\Lambda}$, fix $\mu_{\mathcal{G}}$ s.t. the Weil's formula holds. The measure on Ω is chosen s.t. it is as usual the subspace measure inherited from $\mu_{\mathcal{G}}$. Then the cross-section map $\varsigma : \mathcal{G}/\Lambda \to \Omega$ is a bijective measure-preserving mapping. Furthermore, its inverse $\varsigma^{-1} : \Omega \to \mathcal{G}/\Lambda$, $x \to [x]$ is measureable.*

Proof. Measureability of ς should be clear. Now, take a measureable subset $E \subseteq \Omega$. Clearly $\mu_{\mathcal{G}}(E) \leqslant \mu_{\mathcal{G}}(\Omega) < \infty$. By means of the Weil's formula, and the required choice of Haar measures, we obtain:

$$\begin{aligned}
\mu_\Omega(E) &= \mu_{\mathcal{G}}(E) = \int_{\mathcal{G}} \chi_E(x) \mathrm{d}\mu_{\mathcal{G}}(x) = \int_{\mathcal{G}/\Lambda} \sum_{\lambda \in \Lambda} \chi_E(x + \lambda) \mathrm{d}\mu_{\mathcal{G}/\Lambda}(x + \Lambda) \\
&= \int_{\mathcal{G}/\Lambda} \chi_{\varsigma^{-1}(E)}([x]) \mathrm{d}\mu_{\mathcal{G}/\Lambda}([x]) = \mu_{\mathcal{G}/\Lambda}(\varsigma^{-1}(E)).
\end{aligned}$$

Hence ς is a measure preserving map between $\mathcal{G}/\Lambda$ and Ω.

That ς^{-1} given in the lemma is indeed the inverse of ς is obvious. Since ς^{-1} is the restriction of a continuous mapping (that is the canonical quotient mapping $\mathrm{q} : \mathcal{G} \to \mathcal{G}/\Lambda$), and the restriction of a continuous mapping is always continuous, it follows immediately that ς^{-1} is continuous and hence measureable. So the desired statement holds. ∎

For the cross-transversal map $\tau_{\Omega \to \Omega'}$ between two fundamental domains, we get also a similar result:

Lemma 4.9. *Let $\mathcal{G}$ be an LCA group, Λ be a countable uniform lattice in $\mathcal{G}$, and Ω be a relatively compact fundamental domain of Λ. Given another relatively compact fundamental domain $\Omega^{'}$ of Λ, and consider the cross-transversal map $\tau_{\Omega \to \Omega'}$ between Ω and $\Omega^{'}$. Then $\tau_{\Omega \to \Omega'}$ is a measure-preserving mapping between Ω and $\Omega^{'}$.*

Proof. It is not hard to see that $\tau_{\Omega \to \Omega'} : \Omega \to \Omega^{'}$, $x \mapsto (x + \Lambda) \cap \Omega^{'}$ is measureable. Indeed, let $E \subseteq \Omega^{'}$. It clearly holds $\tau_{\Omega \to \Omega'}^{-1}(E) = (E + \Lambda) \cap \Omega$. Since $E + \Lambda$, as it is a countable union of measureable sets, is a measureable subset of $\mathcal{G}$, it follows immediately by definition, that $(E + \Lambda) \cap \Omega$ is measureable w.r.t. Ω.

Measure preservation property can be shown by computations. For $E \subseteq \Omega^{'}$:

$$\begin{aligned}
\mu_{\Omega'}(E) &= \mu_{\mathcal{G}}(E) = \mu_{\mathcal{G}}(\mathcal{G} \cap E) = \mu_{\mathcal{G}}(\bigcup_{\lambda \in \Lambda} (\Omega + \lambda) \cap E) = \mu_{\mathcal{G}}(\bigcup_{\lambda \in \Lambda} [(E - \lambda) \cap \Omega] + \lambda) \\
&= \mu_{\mathcal{G}}(\tau_{\Omega \to \Omega'}^{-1}(E) + \lambda) = \mu_{\mathcal{G}}(\tau_{\Omega \to \Omega'}^{-1}(E)) \\
&= \mu_\Omega(\tau_{\Omega \to \Omega'}^{-1}(E)).
\end{aligned}$$

As an easy implication of above lemma, one obtains that relatively compact fundamental domains of a uniform lattice has the same measure.

Noticing the fact that cross-section map is a measure preserving map, one immediately obtains the following useful characterization of the Koopman operator related to that map:

Lemma 4.10. *Let $\mathcal{G}$ be an LCA group, Λ be a countable uniform lattice in $\mathcal{G}$. Let Ω be a relatively compact fundamental domain of $\mathcal{G}/\Lambda$ and $\varsigma : \mathcal{G}/\Lambda \to \mathcal{G}$ denotes the cross-section map. Define the mapping $\mathrm{U}_\varsigma(f) = f \circ \varsigma$. Then U_ς is an isometric isomorphism between $L^2(\Omega)$ and $L^2(\mathcal{G}/\Lambda)$. Furthermore, the inverse of U_ς, which is an isometric isomorphism between $L^2(\mathcal{G}/\Lambda)$ and $L^2(\Omega)$, is given by $\mathrm{U}_\varsigma^{-1} f = \tilde{f}$, where $\tilde{f}(x) = f([x])$, for all $x \in \Omega$.*

Proof. To establish the statements, prop. B.5 asserts that it is sufficient to show that ς is a measure-preserving bijective mapping, whose inverse is measureable. Those statements was already shown in lemma 4.8. ∎

To say in a more convenient way, the inverse mapping $\mathrm{U}_\varsigma^{-1}$ can be seen as the restriction of an Λ-periodic function, square-integrable on an relatively compact fundamental domain of $\mathcal{G}/\Lambda$, to the relatively compact fundamental domain Ω of $\mathcal{G}/\Lambda$. Furthermore, the cross-transversal mapping also induces a unitary composition operator:

Lemma 4.11. *Let $\mathcal{G}$ be an LCA group, and Λ be a countable uniform lattice in G. Further, given two fundamental domains Ω and Ω' of Λ. Then the composition operator $\mathrm{U}_{\tau_{\Omega\to\Omega'}}(\cdot) := (\cdot) \circ \tau_{\Omega\to\Omega'}$ is a unitary equivalence between $L^2(\Omega')$ and $L^2(\Omega)$. The corresponding adjoint is given by $\mathrm{U}^*_{\tau_{\Omega\to\Omega'}} = \mathrm{U}_{\tau_{\Omega'\to\Omega}}$.*

In the next section, we shall induce an orthonormal basis for $L^2(\Omega)$ from $L^2(\mathcal{G}/\Lambda)$ by means of U_ς.

4.3. Orthonormal Sampling of Functions in $\mathcal{PW}^2_\Omega$

By previous discussions, it is advantageous to use the following conventions for the Haar measures of the considered LCA groups:

Conventions 4.12. *Let $\mathcal{G}$ be an LCA group, Λ a countable uniform lattice, and Ω a fundamental domain of Λ. If not otherwise stated, we fix from now on the Haar measures μ_Λ s.t. μ_Λ is a counting measure, $\mu_{\mathcal{G}/\Lambda}$ s.t. $\mu_{\mathcal{G}/\Lambda}(\mathcal{G}/K) = 1$. Furthermore, by the previous choices of μ_Λ and $\mu_{\mathcal{G}/\Lambda}$, we fix $\mu_\mathcal{G}$ s.t. the Weil's formula holds. The measure on Ω is as usual the subspace measure inherited from $\mu_\mathcal{G}$.*

Recall that if those conventions are followed, then one can be sure that the following holds:

- $\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)$ forms an ONB for $L^2(\mathcal{G}/\Lambda)$ (Corollary 3.48)
- The cross section map $\varsigma : \mathcal{G}/\Lambda \to \Omega$ is a bijective measure preserving mapping (lemma 4.8).
- The composition operator U_ς is a unitary equivalence between $L^2(\Omega)$ and $L^2(\mathcal{G}/\Lambda)$ (lemma 4.10).

The quintessence of this section lays in the following lemma:

Lemma 4.13. *Let $\mathcal{G}$ be an LCA group, Λ be an admissible uniform lattice in $\mathcal{G}$, and Ω be a relatively compact fundamental domain of Λ. Then the set:*

$$\{\gamma\ \chi_\Omega : \gamma \in \mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)\} \tag{4.1}$$

forms an orthonormal basis for $L^2(\Omega)$.

Proof. We first show the statement for the case that Ω is a fundamental domain of Λ. By definition of uniform lattice, it follows that $\mathcal{G}/\Lambda$ is compact. Hence, by corollary 3.48, the characters of $\mathcal{G}/\Lambda$ forms an orthonormal basis for $L^2(\mathcal{G}/\Lambda)$. Theorem 3.36 asserts that the characters of $\mathcal{G}/\Lambda$ can be identified with those which annihilate Λ, i.e. the set $\{\xi_\gamma : \gamma \in \Lambda^\perp\}$, where $\xi_\gamma([x]) := \gamma(x)$, $\forall x \in \mathcal{G}$.

Consider the inverse composition operator U_ς, induced by the cross-section map ς. We already see in lemma 4.10, that U_ς^{-1} is an unitary operator. In particular U_ς is a linear homeomorphism. Hence from lemma 2.8, $U_\varsigma \xi_\gamma = \gamma\ \chi_\Omega$, $\gamma \in \Lambda^\perp$ forms also an orthonormal basis for $L^2(\Omega)$. So the statement is established for Ω is a fundamental domain. ∎

By noticing that the Lebesgue integral is invariant toward changes in null sets, one obtains the following simple modification of above lemma:

Corollary 4.14. *Let $\mathcal{G}$ be an LCA group, Λ be an admissible uniform lattice in $\mathcal{G}$, and Ω be a relatively compact almost fundamental domain of Λ, or equivalently a tiling set of $\mathcal{G}$ by Λ. Then the set:*

$$\{\gamma\ \chi_\Omega : \gamma \in \mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)\} \tag{4.2}$$

forms an orthonormal basis for $L^2(\Omega)$.

Proof. By the definition of almost fundamental domains, it follows that there exists a fundamental domain $\Omega^{'}$ of Λ which differs with Ω in a null set, hence $L^2(\Omega)$ is isometric isomorph to $L^2(\Omega^{'})$, e.g. by restriction, or continuous extension. Accordingly, the corresponding ONB in $L^2(\Omega^{'})$ as given in lemma 4.13 can be restricted or continuous extended, s.t. they forms an ONB for $L^2(\Omega)$. As annihilators are defined over $\mathcal{G}$, the desired statement follows. ∎

For later purposes, we may generalize above lemma to the direct sum of a collection of such Lebesgue spaces as follows:

Lemma 4.15. *Let Λ be an admissible uniform lattice in an LCA group $\mathcal{G}$. Let D be a (also possible: almost) fundamental domain for Λ. An orthonormal basis for the direct sum of Hilbert spaces $L^2(D)^{\oplus M}$ is $\{\mathbf{e}_\gamma^{(m)} : m \in [M],\ \gamma \in \mathrm{A}_{\mathcal{G}}(\Lambda)\}$, where $\mathbf{e}_\gamma^{(m)} := \boldsymbol{\delta}_m \gamma \chi_D$.*

Proof. This follow immediately from lemma 4.13 and lemma 2.8. ∎

Remark 4.16. *Of course, by the similar argument as given in the proof of 4.17, and by the fact that the direct product of unitary operator is again a unitary operator, the requirement that D is a relatively compact fundamental domain of Λ in lemma 4.15 can be replaced by the requirement that D is a tiling subset of $\mathcal{G}$ by Λ.*

The following lemma, which is basically a modified version of lemma 4.13 may be helpful to established the sampling theorem for LCA groups:

Lemma 4.17. *Let $\mathcal{G}$ be an LCA group, and given an admissible uniform lattice $\tilde{\Lambda}$ in $\mathcal{G}^\wedge$. Given a relatively compact fundamental domain $\tilde{\Omega}$ of $\tilde{\Lambda}$. Choose the Haar measures $\mu_{\mathcal{G}^\wedge}$, μ_Λ, $\mu_{\mathcal{G}/\Lambda}$, and the measure on $\tilde{\Omega}$, by conventions 4.12. Then the following holds:*

$$\{\alpha_{\mathcal{G}}(\lambda)\chi_\Omega : \lambda \in \mathrm{A}_{\mathcal{G}}(\Lambda)\}$$

forms an orthonormal basis for $L^2(\Omega)$.

Proof. By statement (a) in thm. 3.36, the characters $\tilde{\xi}$ on $\mathcal{G}^\wedge/\Lambda$ is induced uniquely by an element $\xi : \mathcal{G}^\wedge \to \mathbb{T}$ of $\mathrm{A}_{\mathcal{G}^{\wedge\wedge}}(\Lambda)$, s.t. $\tilde{\xi}([\gamma]) = \xi(\gamma)$, $\forall \gamma \in \mathcal{G}^\wedge$. In turn, by item (c) in lemma 3.36, each $\xi \in \mathrm{A}_{\mathcal{G}^{\wedge\wedge}}(\Lambda)$ can be identified with a $\lambda \in \mathrm{A}_{\mathcal{G}}(\Lambda)$ by the natural mapping $\alpha_{\mathcal{G}}$, i.e. $\xi = \alpha_{\mathcal{G}}(\lambda)$. So, the dual group of $\mathcal{G}^\wedge/\Lambda$ can be identified with $\mathrm{A}_\mathcal{G}(\Lambda)$, with the identification $\mathrm{A}_\mathcal{G}(\Lambda) \ni \lambda \leftrightarrow \tilde{\xi} \in (\mathcal{G}^\wedge/\Lambda)$, where $\tilde{\xi}([\gamma]) = \alpha_\mathcal{G}(\lambda)(\gamma)$, $\forall \gamma \in \mathcal{G}^\wedge$. Finally, from lemma 4.13, it follows that $\{\alpha_\mathcal{G}(\lambda)\chi_\Omega : \lambda \in \mathrm{A}_\mathcal{G}(\Lambda)\}$ forms an orthonormal basis for $L^2(\Omega)$. ∎

From corr. 4.14, it follows that one can also substitute the requirement that Ω is a fundamental domain of Λ by the requirement that Ω tiles $\mathcal{G}^\wedge$ by Λ.

To construct an orthonormal basis for $\mathcal{PW}^2_\Omega$, we first establish an orthonormal basis for the frequency space of $\mathcal{PW}^2_\Omega$, i.e. $L^2(\Omega)$. In particular, we need an admissible uniform lattice Λ, which fractionizes (up to a null set) the frequency domain $\mathcal{G}^\wedge$ into copies of Ω shifted by elements of Λ. So, it is necessary to restrict our consideration from general LCA groups to compactly generated LCA groups, since in this class of groups, the existence of a uniform lattice is ensured. Thm. 3.16 asserts that the condition that $\mathcal{G}^\wedge$ is compactly generated topological group is sufficient for the existence of a countable uniform lattice. As we shall see in the following lemma, this restriction has no serious affect to our consideration:

Lemma 4.18. *Let $\mathcal{G}$ be an LCA group, $\Omega \subseteq \mathcal{G}^\wedge$ relatively compact. Generate the open subgroup H of $\mathcal{G}^\wedge$ by Ω, i.e. $H := \bigcup_{k\in\mathbb{N}} k\Omega$. Then there exists a compact subgroup K of $\mathcal{G}$ (e.g. $K = \mathrm{A}_\mathcal{G}(H)$) s.t. every $f \in \mathcal{PW}_\Omega$ is K-periodic.*

Proof. Take an open relatively compact set $V \subseteq \mathcal{G}^\wedge$, which contains Ω. Define $H = \bigcup_{k\in\mathbb{N}} kV$. Notice that H is an open subgroup of $\mathcal{G}^\wedge$, and $\mathcal{G}/H$ is discrete. By the latter statement, it follows that $(\mathcal{G}^\wedge/H)^\wedge$ is compact. Now define $K := \mathrm{A}_\mathcal{G}(H)$. For $f \in \mathcal{PW}^2_\Omega$, $x \in \mathcal{G}$, and $k \in K$, we compute:

$$f(x+k) = \int_\Omega \hat{f}(\gamma)\gamma(x+k)\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma) = \int_\Omega \hat{f}(\gamma)\gamma(x)\gamma(k)\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma) = \int_\Omega \hat{f}(\gamma)\gamma(x)\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma) = f(x), \qquad ∎$$

where the first equality follows from the inversion theorem and band-limitedness of f, the second from the fact that characters are homomorphism, and the third from the fact that k is the annihilator of Ω, which shows the desired statement. The fact that K is open, follows from the following observation: From item (a) in thm. 3.36, it follows that $\mathrm{A}_{\mathcal{G}^{\wedge\wedge}}(H) \cong (\mathcal{G}^\wedge/H)^\wedge$. Furthermore, from item (c) in prop. C.9, we know that $\mathcal{G}^\wedge/H$ is discrete, since H is open, and accordingly, $(\mathcal{G}^\wedge/H)^\wedge$ is compact, since the dual of a discrete set is compact. Pontryagin-van Kampen duality give the remaining hint.

According to this lemma, all square-integrable signals, band-limited to a relatively compact subset Ω of $\mathcal{G}^\wedge$ lives essentially in $L^2(\mathcal{G}/K)$, i.e. it is K-periodic. So, the engineer's task is to guess the possible band limit K of the signal of interests,

Now, we are ready to give the desired sampling Theorem for signals in $\mathcal{PW}^2_\Omega$:

Theorem 4.19 (Kluvanek [31]). *Let $\mathcal{G}$ be an LCA group. Given an $f \in \mathcal{PW}^2_\Omega$, where $\Omega \subseteq \mathcal{G}^\wedge$ is a relatively compact measureable subset. Assume that there exists an admissible uniform lattice Λ in $\mathcal{G}^\wedge$ (e.g. in case $\mathcal{G}^\wedge$ is second-countable/metrizable), for which Ω is a fundamental domain. Further, choose the Haar measures $\mu_{\mathcal{G}^\wedge}$, μ_Λ, $\mu_{\mathcal{G}/\Lambda}$, and the measure on $\tilde{\Omega}$, by conventions 4.12, and subsequently, choose $\mu_\mathcal{G}$ s.t. the Fourier inversion formula holds. Then:*

$$\tilde{f} = \sum_{\lambda\in\Lambda^\perp} f(\lambda)\phi((\cdot)-\lambda), \quad \text{a.e. } x \in \mathcal{G},$$

where $\phi(\cdot) = \widecheck{\chi_\Omega}(\cdot)$, defines a continuous function in $L^2(\mathcal{G})$ and it holds:

$$f(x) = \tilde{f}(x) \quad \text{a.e. } x \in \mathcal{G}.$$

Proof. From lemma 4.17, we know that $\{\alpha_\mathcal{G}(\lambda)\chi_\Omega : \lambda \in \mathrm{A}_\mathcal{G}(\Lambda)\}$ forms an orthonormal basis for $L^2(\Omega)$. We already know that the inverse of the Fourier transform is a unitary equivalence between $L^2(\Omega)$ seen as a closed subspace of $L^2(\mathcal{G}^\wedge)$ and $\mathcal{PW}^2_\Omega$, and in particular, it preserves orthonormal bases. Hence $\{\widecheck{\chi_\omega}(\cdot+\lambda)\}_{\lambda\in \mathrm{A}_{\mathcal{G}}(\Lambda)}$ form an ONB for $\mathcal{PW}^2_\Omega$, since $\widecheck{\chi_\omega}(\cdot+\lambda)$ is the image of $\alpha_{\mathcal{G}}(\lambda)\chi_\Omega$ under the inverse Fourier transform. Furthermore, given $\hat{f}\in L^2(\Omega)$, and let $\hat{f}=\sum_{\lambda\in \mathrm{A}_{\mathcal{G}}(\Lambda)} c_\lambda \alpha_{\mathcal{G}}(\lambda)\chi_\Omega$ be the corresponding orthonormal expansion of $\hat{f}$, where $c_\lambda = \langle \hat{f}, \alpha_{\mathcal{G}}(\lambda)\rangle_{L^2(\Omega)}$. It follows immediately that f has the expansion:

$$f=\sum_{\lambda\in \mathrm{A}_{\mathcal{G}(\Lambda)}} c_\lambda \widecheck{\chi_\omega}(\cdot+\lambda), \tag{4.3}$$

where the equality is to understand as a.e. and the convergence has to be understand in the $\mathcal{PW}^2_\Omega$-sense, hence in the $L^2(\mathcal{G})$-sense. Now we need to show that $\{c_\lambda\}_{\lambda\in \mathrm{A}_{\mathcal{G}}(\Lambda)}$ are the samples of f taken at points of Λ. Indeed, by computations for each $\lambda\in A_{\mathcal{G}}(\Lambda)$:

$$\langle \hat{f}, \alpha_{\mathcal{G}}(\lambda)\chi_\Omega\rangle_{L^2(\Omega)} = \int\limits_\Omega \hat{f}(\gamma)\overline{\gamma(\lambda)\chi_\Omega}(\gamma)\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma) = \int\limits_{\mathcal{G}^\wedge} \hat{f}(\gamma)\overline{\gamma(\lambda)}\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma) = f(-\lambda),$$

where the second inequality follows from the fact that $\hat{f}$ is supported in Ω, and the third from the definition of the inverse Fourier transform. Setting this result to the expansion of f, and by the permutation of the sum, and noticing that ONB is an unconditional basis, one obtains that:

$$f=\sum_{\lambda\in \mathrm{A}_{\mathcal{G}}(\Lambda)} f(\lambda)\widecheck{\chi_\omega}(\cdot-\lambda),$$

in L^2-sense

Since $\hat{f}$ is contained in $L^2(\Omega)$, it follows that $\hat{f}$ is also contained in $L^1(\Omega)$, and accordingly in $L^1(\mathcal{G}^\wedge)$. Hence the inverse Fourier transform of $\hat{f}$ yields an $\tilde{f}\in C_0(\mathcal{G})$. By continuity of the inverse Fourier transform and the convergence of the basis expansion of $\hat{f}$ in $L^1(\mathcal{G}^\wedge)$-norm[2], it follows that:

$$\tilde{f}=\sum_{\lambda\in \mathrm{A}_{\mathcal{G}(\Lambda)}} c_\lambda \widecheck{\chi_\omega}(\cdot+\lambda), \tag{4.4}$$

where the convergence is w.r.t. the supremum norm (notice that the supremum norm in C_0 is related to sup and not just essentially supremum), and accordingly is uniform. Clearly $f=\tilde{f}$ a.e. on $\mathcal{G}$. ∎

Of ancilliary interests, one can give a hint about the shape of the kernel ϕ used in the orthonormal sampling expansions for $\mathcal{PW}^2_\Omega$:

Proposition 4.20. *Let $\mathcal{G}$ be an LCA group, and $\tilde{\Lambda}$ be an admissible uniform lattice in $\mathcal{G}^\wedge$, for which a relatively compact measureable subset $\Omega\subseteq\mathcal{G}^\wedge$ is a fundamental domain. Let $\phi:\mathcal{G}\to\mathbb{C}$ be a function on an LCA group be given by the expression:*

$$\phi(x)=\int\limits_\Omega \gamma(x)\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma), \quad x\in\mathcal{G}. \tag{4.5}$$

Then ϕ is defined everywhere on $\mathcal{G}$. In particular, it is a continuous function vanish at infinity, positive-definite, belongs to $L^2(\mathcal{G})$, and its $L^2(\mathcal{G})$-norm is equal to 1. Furthermore, the values of ϕ on $\mathrm{A}_{\mathcal{G}}(\Lambda)$ can be given by:

$$\phi(\lambda)=\begin{cases}1, & \text{if } \lambda=0_{\mathcal{G}}\\ 0, & \text{if } \lambda\in \mathrm{A}_{\mathcal{G}}(\Lambda)\backslash\{0_{\mathcal{G}}\}\end{cases}. \tag{4.6}$$

Proof. The fact that ϕ is defined everywhere on $\mathcal{G}$, that $\phi\in C_0(\mathcal{G})$, and that it is an element of $L^2(\mathcal{G})$ with norm 1, is mentioned implicitly in the proof of Thm. 4.19. As a direct implication of Bochner's

[2] convergence in $L^2(\Omega)$-norm $\Rightarrow$ convergence in $L^1(\Omega)$-norm $\Rightarrow$ convergence in $L^1(\mathcal{G}^{\wedge\wedge})$-norm

Theorem, ϕ is positive-definite. By considering an alternative description of ϕ:

$$\phi(x) = \int_{\Omega} \alpha_{\mathcal{G}}(x) \mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma), \quad x \in \mathcal{G},$$

and since $\{\alpha_{\mathcal{G}}(\lambda)\chi_\Omega : \forall \lambda \in \mathrm{A}_{\mathcal{G}}(\Lambda)\}$ forms an ONB for $L^2(\Omega)$, (4.6) holds. ∎

Remark 4.21. *From the positive-definiteness of ϕ, ϕ fulfills the property given in 3.43. The property given in 3.43 and the property (4.6) remind one of the sinc-kernel which occurs in the WKS-formula.*

The following alternative version of above Thm. may also be helpful for later approach:

Corollary 4.22 (Kluvanek's Sampling Formula for frequency sampling). *Let $\mathcal{G}$ be a compactly generated LCA group, $\Omega \subseteq \mathcal{G}$ be a relatively compact subset. Assume that there exists an admissible lattice Λ in $\mathcal{G}$, for which Ω is a fundamental domain. Follow the conventions 4.12, and fix $\mu_{\mathcal{G}^\wedge}$ s.t. the Fourier inversion formula holds. Then each $f \in \tilde{\mathcal{PW}}_\Omega$ is equal a.e. to a function $\tilde{f} \in C_0(\mathcal{G}^\wedge)$, which is given by:*

$$\tilde{f} = \sum_{\lambda \in \mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)} f(\lambda)\phi((\cdot)\lambda^{-1}),$$

where $\phi = \mathcal{F}(\chi_\Omega)$.

Proof. Similar to the proof of Thm. 4.19, consider the ONB $\{\lambda\chi_\Omega : \lambda \in \mathrm{A}_{\mathcal{G}^\wedge}\}$ of $L^2(\Omega)$. Expanding $\check{f}$ by means of that ONB, and by computing the inner product between f and each $\lambda\chi_\Omega$, for all $\lambda \in \mathrm{A}_{\mathcal{G}^\wedge}$, one obtains the series:

$$\check{f} = \sum_{\lambda \in \mathrm{A}_{\mathcal{G}^\wedge(\Lambda)}(\Lambda)} f(\lambda)\ \lambda(\cdot), \tag{4.7}$$

where the convergence is in $L^2(\Omega)$-sense, $L^1(\Omega)$-sense, and accordingly in $L^1(\mathcal{G})$-sense. Fourier transforming above expression, one obtains immediately the desired statements by similar arguments given in the proof of Thm. 4.19. ∎

4.4. Orthonormal Sampling in Euclidean Spaces by Regular Lattices

To visualize the idea introduced previously in the previous section, we shall apply in this section Thm. 4.19 to finite-energy band-limited functions defined on the euclidean spaces. First, it is necessary to classify uniform lattices in Euclidean spaces:

4.4.1. Uniform Lattices in $\mathbb{R}^N$ - (Full-Rank) Lattices

It is clear that $\mathbb{R}^N$ equipped with the natural topology is second countable (observe that for a $x \in \mathbb{R}^N$, $\{\mathcal{B}_{\frac{1}{n}}(x)\}_{n\in\mathbb{N}}$ forms a countable neighborhood base of x). Recall that any subspace with corresponding subspace topology of a second countable space is again second countable, and any discrete second countable space is countable. Hence it follows immediately that any discrete subgroup of $\mathbb{R}^N$ is countable, thus it is unnecessary to make a distinction between discrete - and countable subgroup.

Obviously, a discrete subgroups Λ of $\mathbb{R}^N$ are exactly those subgroups which can be generated minimally by finite elements of $\mathbb{R}^N$, i.e. Λ can be written by $\Lambda = \langle a_1, \dots, a_k\rangle$, for finite K, and $a_1, \dots, a_K \in \mathbb{R}^N$. Let be the *rank of a subgroup* is defined as the smallest cardinality of the generating set. The following proposition sheds light on the structure of uniform lattices in the euclidean space $\mathbb{R}^N$:

Lemma 4.23. *Given a discrete subgroup Λ of $\mathbb{R}^N$. Λ is a uniform lattice, if and only if Λ is of full-rank.*

Proof. "$\Leftarrow$": Let Λ be generated by N linearly independent vectors $a_1, \dots, a_N \in \mathbb{R}^N$, and let A be the non-singular $N \times N$ real matrix, whose columns are exactly the generator of Λ, i.e. $A := [a_1, \dots, a_N]$. Consider the mapping $f_{A^{-1}} : \mathbb{R}^N \to \mathbb{R}^N$, $x \mapsto A^{-1}x$. It is obvious that $f_{A^{-1}}$ is a topological group isomorphism between $\mathbb{R}^N$ and $\mathbb{R}^N$. From the fact that quotient groups are invariant under topological isomorphism (corr. C.12), we obtain: $\mathbb{R}^N/\Lambda \cong f_{A^{-1}}(\mathbb{R}^N)/f_{A^{-1}}(\Lambda) = \mathbb{R}^N/\mathbb{Z}^N$. Since $\mathbb{R}^N/\mathbb{Z}^N \cong (\mathbb{R}/\mathbb{Z})^N$, and clearly $(\mathbb{R}/\mathbb{Z})^N \cong \mathbb{T}^N$, it follows immediately that $\mathbb{R}^N/\Lambda$ is topological group isomorphic to $\mathbb{T}^N$, and hence compact.

"$\Rightarrow$": We proof this implication by contradiction. Suppose that Λ is generated by $M < N$ linearly independent vectors $a_1, \dots, a_M$ in $\mathbb{R}^N$. Take an $x \in \mathbb{R}^N \backslash \operatorname{span}\{a_1, \dots, a_M\}$. Consider the restriction $\mathrm{q}|_{\operatorname{span}\{x\}} : \operatorname{span}\{x\} \to \mathbb{R}^N/\Lambda$, of the canonical quotient mapping $\mathrm{q} : \mathbb{R}^N \to \mathbb{R}^N/\lambda$. Now, we aim to show that $\mathrm{q}|_{\operatorname{span}\{x\}}$ is injective. Let $0_{\mathbb{R}^N/\Lambda}$ be the identity in $\mathbb{R}^N/\lambda$. Clearly, $\mathrm{q}|_{\operatorname{span}\{x\}}{}^{-1}(0_{\mathbb{R}^N/\lambda}) = \Lambda \cap \operatorname{span}\{x\}$. Since x is, by choice, not contained in $\operatorname{span}\{a_1, \dots, a_M\}$, $\Lambda \cap \operatorname{span}\{x\}$ contains only the singleton $\{0\}$. Hence, $\mathrm{q}|_{\operatorname{span}\{x\}}$ is injective. It is not hard to show that $\mathrm{q}|_{\operatorname{span}\{x\}}(\operatorname{span}\{x\})$ is not compact in $\mathbb{R}^N/\Lambda$, and by this reason, $\mathbb{R}^N/\Lambda$ can not be compact. ∎

In literatures, such discrete subgroup is called *(full-rank) lattice* in $\mathbb{R}^N$. Such a subgroup Λ can also be written as $\Lambda = A\mathbb{Z}^N$, where $A \in \mathbb{R}^{N \times N}$ is a non-singular matrix. We call such a matrix A *basis of* Λ, and say Λ *is generated by* A. Let $\{a_k\}_{k \in [N]}$ be the column vector of A, in some cases, we use the term: $\{a_k\}_{k \in [N]}$ *is a basis for* Λ, and Λ *is generated by* $\{a_k\}_{k \in [N]}$.

Notice that basis of a lattice is not necessarily unique. Recall that a unimodular matrix $U \in \mathbb{R}^{N \times N}$ is defined as a matrix with integer entries, whose determinant has the modulus ± 1. It is not hard to show, that the following relationship between two bases A and B of a uniform lattice Λ hold: $A = BU$, where U is unimodular. Let Λ be a lattice which is generated by A. One can denote Λ_A instead of Λ to emphasize the generating system associated with this lattice (or equivalently $\Lambda(a_1, \dots, a_N)$, where $a_1, \dots, a_N$ are the column vectors of A).

To a lattice Λ_A, one associates a parallepiped $\Phi(\Lambda_A)$ of $\mathbb{R}^N$, which is defined as $\Phi_{\Lambda_A} := A[0,1)^n$, called *fundamental parallepiped*. We write also $\Phi(A)$, and $\Phi(a_1, \dots, a_N)$. It is obvious that $\Phi(A)$ is a relatively compact measureable transversal of $\mathbb{R}^N/\Lambda$. Furthermore, subsets of $\mathbb{R}^N$, which are Λ-equivalent to $\Phi(A)$ are clearly relatively compact measureable transversal of $\mathbb{R}^N/\Lambda$. It can easily be shown that the volume of $\Phi(A)$ is connected with the matrix A, in the sense that $m(\Phi(A)) = |\det A|$. We have seen in lemma 4.8, that all relatively compact fundamental domains coincides in measure. Hence, all relatively compact fundamental domains of Λ have the measure $|\det A|$. To visualize the concept of lattices and Fundamental parallepiped, two examples of lattices in the planar domain $\mathbb{R}^2$ with corresponding fundamental parallepipeds are depicted in fig. 4.1.

4.4.2. Annihilator of Uniform Lattices in $\mathbb{R}^N$ - Dual Lattices

Recall that the dual group of $\mathbb{R}^N$ can be identified with $\mathbb{R}^N$ by the identification $(\mathbb{R}^N)^\wedge \ni \gamma_\omega = e^{2\pi i \langle \cdot, \omega \rangle} \leftrightarrow \omega$, i.e. the characters of $\mathbb{R}^N$ are exactly the exponential functions of the form $e^{2\pi i \langle \omega, \cdot \rangle}$, indexed by $\omega \in \mathbb{R}^N$. Given a uniform lattice Λ in $\mathbb{R}^N$. Obviously, the annihilator of the uniform lattice Λ in $(\mathbb{R}^N)^\wedge$ can be identified with the subset $\{\lambda' \in \mathbb{R}^N : \langle \lambda', \lambda \rangle \in \mathbb{Z},\ \forall \lambda \in \Lambda\}$ of $\mathbb{R}^N$, since the restriction of a topological group isomorphism to a subgroup is also a topological group isomorphism to its image. This set plays an important role in the lattice theory in euclidean spaces:

Definition 4.9 (Dual Lattice). *Let Λ be a full-rank lattice. The set $\Lambda^\perp$ corresponding to Λ, which is defined as:*

$$\{x \in \mathbb{R}^N : \langle x, \lambda \rangle \in \mathbb{Z},\ \forall \lambda \in \Lambda\}, \tag{4.8}$$

is called dual lattice of Λ.

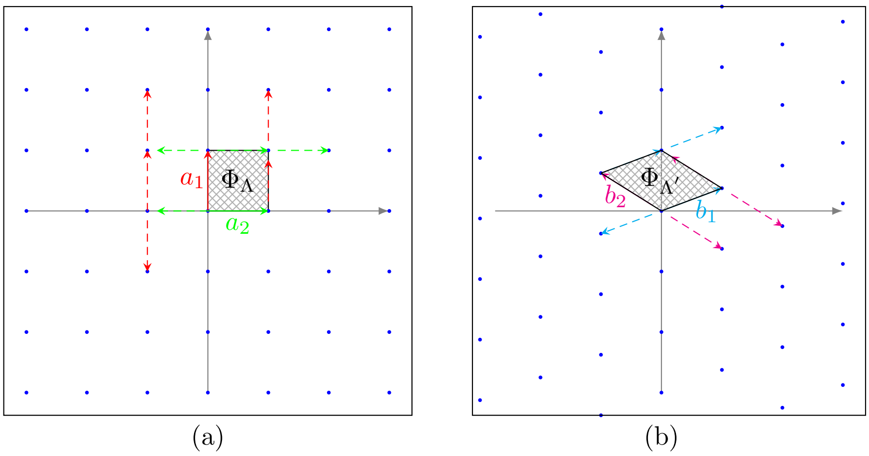


Figure 4.1.: Two examples of a lattice in $\mathbb{R}^2$: (a) Lattice Λ with basis $\{a_1, a_2\}$ and fundamental domain Φ_Λ, (b) Lattice $\Lambda^{'}$ with basis $\{b_1, b_2\}$ and fundamental domain $\Phi_{\Lambda'}$

Define for a set $X \subseteq \mathbb{R}^N$, we denote: $\mathcal{E}(X) := \{e^{2\pi i x(\cdot)} : x \in X\}$. So the relationship between the annihilator $\mathrm{A}_{(\mathbb{R}^N)^\wedge}(\Lambda)$ and the dual lattice $\Lambda^\perp$, can now be described by the equality $\mathrm{A}_{(\mathbb{R}^N)^\wedge}(\Lambda) = \mathcal{E}(\Lambda^\perp)$. The dual lattice of a full-rank lattice $\Lambda \subseteq \mathbb{R}^N$ can easily be computed, in case that the generator of Λ is given. Let $A \in \mathbb{R}^{N\times N}$ be a generator of Λ. We already show in lemma 4.1 that the annihilator of Λ is a uniform lattice in $\mathcal{G}^\wedge$. So it follows from the fact annihilators and dual lattices are identifiable, and $(\mathbb{R}^N)^\wedge$ and $\mathbb{R}^N$ are identifiable, that the dual lattice $\Lambda^\perp$ is also a uniform lattice in $\mathbb{R}^N$. By lemma 4.23, $\lambda^\perp$ is generated by a non-singular matrix. Furthermore, it is immediate to see that $\Lambda^\perp$ is unique, since $\mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)$ is by definition unique. Accordingly, different representations of $\Lambda^\perp$ by different non-singular matrices differs only in a unimodular matrix. One can easily convince oneself that the lattice, which is generated by the non-singular matrix $B = (A^\mathrm{T})^{-1}$ fulfills (4.8). By this reason, we can state the following lemma:

Lemma 4.24. *Let Λ be a lattice in $\mathbb{R}^N$, generated by a non-singular matrix $A \in \mathbb{R}^{N\times N}$. Then the annihilator of Λ can be identified with the lattice of the form $\Lambda^\perp = (A^\mathrm{T})^{-1}U\mathbb{Z}^N$, where $U \in \mathbb{R}^{N\times N}$ is a unimodular matrix.*

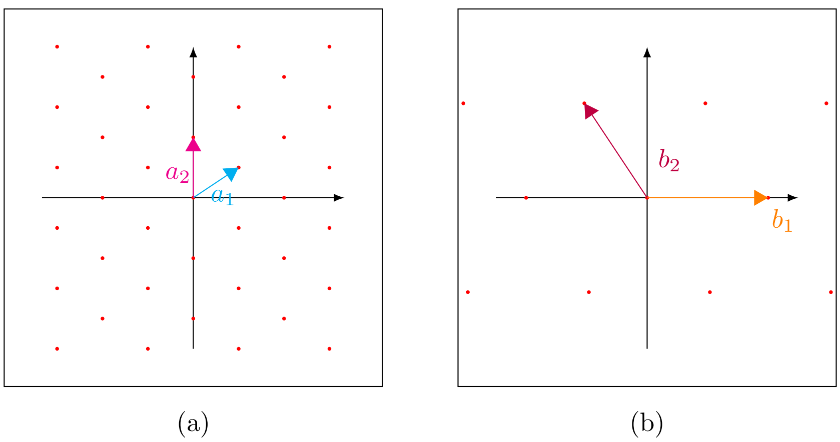


Figure 4.2.: (a) an example of a lattice and (b) its corresponding dual lattice

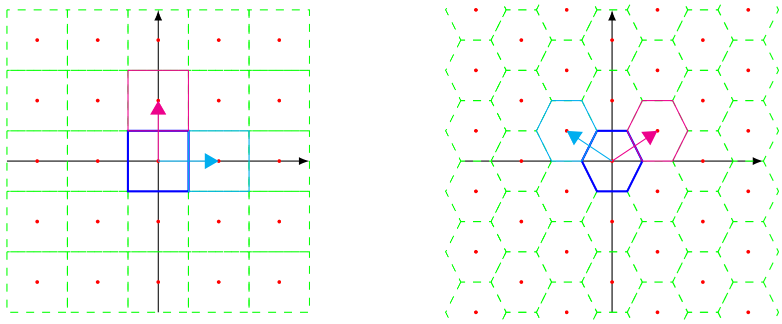


Figure 4.3.: Two different ways to tile $\mathbb{R}^2$

4.4.3. Shape of Fundamental Domains of Uniform Lattices in $\mathbb{R}^N$

We first gives the terms which is used by Fuglede in his work. A discrete subset Λ of $\mathbb{R}^N$ is said to be a spectrum for Ω if $\mathcal{E}(\Lambda)$ is an ONB for $L^2(\Omega)$. Let Λ be a spectrum, in case that Λ is in addition, then we say Λ is a lattice spectrum. A domain in $\mathbb{R}^N$, which possessess a spectrum is called spectral set. Fuglede conjectured in his work [15], that a domain Ω in $\mathbb{R}^n$ admits a spectrum if and only if it is possible to tile $\mathbb{R}^N$ by a family of translates (i.e. a discrete set in $\mathbb{R}^N$) of Ω. He proved the conjecture under the restriction that the tiling set is a lattice in $\mathbb{R}^N$:

Theorem 4.25 (Fuglede). *Let $\Omega \subset \mathbb{R}^N$ be measureable, Λ a lattice, and Λ^* its dual. Ω tiles $\mathbb{R}^N$ with translation set Λ if and only if Λ^* is a spectrum for Ω*

This is exactly a special case of lemma 4.13.

Recently, some facts about the geometrical structure of connected sets admitting a spectrum have been proved. Of course, the restrictions holds also for domains of our interests, i.e. domains which admits a lattice spectrum.

- A ball is not a spectral set in any dimension > 1 [26]. The case in 2 dimension was already shown by Fuglede
- A non-symmetric convex body does not admit a spectrum [32]
- A symmetric convex body in $\mathbb{R}^N$, $n \geqslant 2$, whose boundary is smooth, is not a spectral set [27]
- A set in $\mathbb{R}^2$ admits a spectrum if and only if it is quadrilateral or hexagon [28].

Without giving any comments, we state in the following the structure of lattice tiling sets in general $\mathbb{R}^N$:

Theorem 4.26. *Let $\Omega \subseteq \mathbb{R}^N$ be a non-empty compact convex subset. Then Ω tiles $\mathbb{R}^N$ by a lattice if and only if:*

(a) Ω is a convex polytope.

(b) Ω is centrally symmetric.

(c) Each facet of Ω is centrally symmetric.

(d) Each belt of Ω consists of 4 o 6 facets.

For detailed treatment of above Thm. see e.g. Thm. 32.2 in [20].

4.4.4. Reconstruction formula for Band-Limited Signals in $\mathbb{R}^N$

Let Λ be a uniform lattice in the frequency domain $(\mathbb{R}^N)^\wedge = \mathbb{R}^N$ (we write anyway $(\mathbb{R}^N)^\wedge$, even if we mean $\mathbb{R}^N$), and $\Omega \subseteq (\mathbb{R}^N)^\wedge$ be a relatively compact fundamental domain of this lattice. The Haar measure on $(\mathbb{R}^N)^\wedge/\Lambda$ is chosen by identifying each elements of $\mathbb{R}^N/\Lambda$ with elements of Ω in the usual manner, i.e. $\mathbb{R}^N/\Lambda \ni [x] \leftrightarrow x \in \Omega$ and subsequently by taking the normalized Haar measure on Ω, i.e.:

$$\mu_{(\mathbb{R}^N)^\wedge/\Lambda}([E]) = \frac{1}{m(\Omega)} m(E), \quad \forall [E] \text{ measureable subset of } (\mathbb{R}^N)^\wedge/\Lambda, \tag{4.9}$$

where m denotes the usual Lebesgue-Borel measure in $\mathbb{R}^N$. μ_Λ is chosen as the counting measure. The corresponding Haar measure $\mu_{(\mathbb{R}^N)^\wedge}$ is chosen as m, normalized by $m(\Omega)$. One can immediately convince oneselves that by this choices, the Weil's formula holds. Now it remains tho choose $\mu_{\mathbb{R}^N}$ s.t. the Fourier inversion formula holds. The only choice isclearly: $\mu_{\mathbb{R}^N}(\cdot) = m(\Omega) m(\cdot)$.

Once we have fixed the Haar measures, we can apply the Kluvanek's sampling Thm. (thm. 4.19) to the Euclidean space:

Corollary 4.27 (WKS-Formula in $\mathbb{R}^N$). *Let Λ be a uniform lattice in the frequency domain $(\mathbb{R}^N)^\wedge = \mathbb{R}^N$, and $\Omega \subseteq (\mathbb{R}^N)^\wedge$ be a relatively compact almost fundamental domain of Λ. Each $f \in (\mathcal{PW}^2_\Omega \cap C(\mathbb{R}^N))$ can be reconstructed by the formula:*

$$f = \sum_{\lambda \in \Lambda^\perp} f(\lambda)\phi((\cdot) - \lambda), \tag{4.10}$$

where ϕ is given by:

$$\phi(x) := \frac{1}{m(\Omega)} \int_{\mathbb{R}^N} e^{2\pi i x \omega} \mathrm{d}m(\omega), \quad x \in \mathbb{R}^N,$$

For the approach made in the next chapter we need an alternative version of above formula. But first, we need to take another choice of the Haar measures. Let Λ be a uniform lattice in $\mathbb{R}^N$, and $\Omega \subset \mathbb{R}^N$ be a relatively compact fundamental domain of Λ. The Haar measure on $\mathbb{R}^N/\Lambda$ is chosen as similar as (4.9). The Haar measure on Λ is simply the counting measure. The Haar measure on $\mathcal{G}$ is chosen $\mu_{\mathbb{R}^N} = m(\cdot)/m(\Omega)$. Clearly Weil's formula is fulfilled. Furthermore, an appropriate choice of Haar measure of $(\mathbb{R}^N)^\wedge$ is $\mu_{(\mathbb{R}^N)^\wedge} = m(\Omega) m(\cdot)$. By those choices of Haar measures, and by applying corollary 4.22, we obtain the following alternative version:

Corollary 4.28. *Let Λ be a uniform lattice in $\mathbb{R}^N$, and $\tilde{\Omega} \subset \mathbb{R}^N$ be a relatively compact almost fundamental domain of Λ. Each $f \in \tilde{\mathcal{PW}}_{\tilde{\Omega}}$ can be reconstructed by the formula:*

$$f = \sum_{\lambda \in \Lambda^\perp} f(\lambda)\phi((\cdot) - \lambda), \tag{4.11}$$

where ϕ is given by:

$$\phi(\omega) := \frac{1}{m(\Omega)} \int_{\mathbb{R}^N} e^{-2\pi i x \omega} \mathrm{d}m(x), \quad \omega \in \mathbb{R}^N,$$

5. Application: Phaseless Recovery of Signals on $\mathbb{R}^2$ by Structured Modulations

The problem of recovering a signal from the magnitude of its Fourier transform, which is also called *phase retrieval*, emerges in several applications, e.g. in diffraction imaging applications such as X-ray crystallography, astronomical imaging or speech processing. The phase retrieval problem originates from the physical nature of sensors: since sensors can only record the intensity of the incoming signal, which is generally available in the form of complex valued function, its phase is lost. The phase of a signal is in general independent from its amplitude.

Some approaches to overcome this disbenefit, which adress to finite dimensional signal, have been made. For example, one can use prior knowledge of the signal, such as band-limitation, causality, or certain sufficient conditions on the z-transform of the signal, to recover the signal from its magnitude [21, 50]. In the case of non-availability of preceding - or availability of little preceding knowledge of the object of interest, one can proceed several measurements on this object by slightly different conditions, such as distorted-object method [59], aperture-plane modulation method [11, 63], or fractional Fourier transform method [29]. The signal can be recovered from different measurements by performing iterative alternating projection algorithms. Although, the convergence of this approach strongly depends on specific signal constraint. An analytical frame theoretical solution of finite dimensional phase retrieval problem was delivered by Balan et al. [2, 3]. It will later be stated explicitly in a short manner, since it provide a foundation for the approach made in this work.

It is natural to ask, how to handle the phase retrieval problem for infinite dimensional signal: What is the minimal sampling rate which leads to a perfect reconstruction of a signal from the infinite samples of its magnitude, and how such a sampling and reconstruction scheme, which achieves that corresponding rate, should be. It was shown in [53] that one may sufficiently take the samples of the magnitude of a real-valued signal at twice of the Nyquist rate. Unfortunately, this approach can not be extended to complex valued signals. Moreover, results from frame theoretical approach for the phase retrieval problem in finite dimension delivered in [2, 3] indicate that oversampling alone may not be sufficient to guarantee a perfect reconstruction of the signal of interest. There, particular choice of measurement vectors was the key to enabling signal recovery.

A first attempt on phase retrieval of complex-valued L^2-signals of 1 variable with finite support was made in [60, 48]. There, it was shown, that perfect recovery is guaranteed, if amplitude measurements taken by specific setting at four times the Nyquist rate is given. It provides a reconstruction algorithm, which includes the ideas introduced in [2], and which is inspired by the structured modulation measurements frequently used in optics [7, 11, 59]. A corresponding simulation for finite dimensional signal is available in [48].

The present chapter provides an extension of previous method to signals with finite energy on $\mathbb{R}^2$. Analogue to the case of one variable, the suggested recovery procedure involves three main steps: first, modulation of the signal by a bank of mask functions of appropriate choice and subsequent sampling of the intensity of the modulated signals in the Fourier domain, and second, recovery of the samples of the signal of interest from the samples of its modulated version and subsequent interpolation of the signal. The latter step presupposes blocks containing finite number of intensity samples of the signal of interest, which resulted by applying finite dimensional phase recovery algorithm, introduced in [2, 3], to the samples of the modulated versions of the signal. By ensuring overlapping between subsequent blocks, the unimodular factors in all blocks are matched. So, finally, the well-known interpolation theorem and the inverse Fourier transform can be used to recover the planar domain signal.

By a particular application, it will be shown that for band-limited signals on planar domain (up to some exceptions), and by appropriate specific choice of the sampling lattice, that 8 times the Nyquist rate, instead of the expected 16 times the Nyquist rate, is sufficient to ensure perfect recovery of the signal.

Before we go into detail, we first give in the following section an auxiliary result on phase retrieval of signals on finite sets which is due to [2, 3].

5.1. Finite Dimensional Phase Retrieval

Finite-dimensional phase retrieval problem can be expressed as follows: Given a finite set of measurements $\{c_m\}_{m\in[M]}$ of a finite dimensional signal $x \in \mathbb{C}^M$. Reconstruct x from the available information in form of $\{c_m\}_{m\in[M]}$, where $\{c_m\}_{m\in[M]}$ is given as the square modulus of the inner product of x with appropriately chosen ensemble of vectors $\{a^m\}_{m\in[M]}$ in $\mathbb{C}^N$, i.e. $c_m = |\langle a^m, x\rangle|^2$, $m \in [M]$.

As we will see soon an appropriate choice of measurements vectors is the so called maximal 2-uniform M/K-tight frame, which is defined in the following

Definition 5.1 (Maximal 2-uniform M/K-tight frame). *A set of vectors $\{a^{(m)}\}_{m=1}^M$ in $\mathbb{C}^K$ is said to be a 2-uniform A-tight frame for $\mathbb{C}^K$ if there exist $A, B > 0$ s.t.:*

$$A\|x\|_2^2 \leqslant \sum_{m=1}^{M} |\langle x, a^{(m)}\rangle|^2 \leqslant B\|x\|_2^2 \quad \forall x \in \mathbb{C}^K \text{ (A-tight frame)},$$

with $A = B$, and if for all $j \neq k$:

$$|\langle a_j, a_k\rangle| = c \text{ (2-uniform A-tight frame)},$$

and the number of vectors $M = K^2$ (maximal 2-uniform M/K-tight frame).

The following Thm. ensures the restorability of finite-dimensional signals from the modulus of its measurements:

Theorem 5.1 (Balan, Bodemann, Casazza, Edidin). *Let $\{a^{(m)}\}_{m=1}^M \subset \mathbb{C}^K$ be a 2-uniform M/K tight frame, and $Q_y := yy^*$, $y \in \mathbb{C}^K$. For any $x \in \mathbb{C}^K$:*

$$Q_x = \frac{K(K+1)}{M} \sum_{m=1}^{M} c^{(m)} Q_{a^{(m)}} - c^{(m)}\, Id,$$

where $c^{(m)} = |\langle a^{(m)}, x\rangle|^2$, $\quad m = 1, 2, ..., M$ and Id denotes the identity in $\mathbb{C}^{K\times K}$.

Constructions of 2-uniform M/K-tight frames with $M = K^2$ vectors for different dimensions K can be found in [62]. Later, we shall use the construction for dimension $K = 4$ given in that work.

Pedarsani et. al. gives in [45] an algorithm which provides a reconstruction of vectors in $\mathbb{C}^N$ by only $3N$ measurements, which is considerably lesser than the number of measurements provided from Thm. 5.1. However, the behaviour of the algorithm under presence of noise, given in [45] might be worser than the algorithm given in Thm. 5.1.

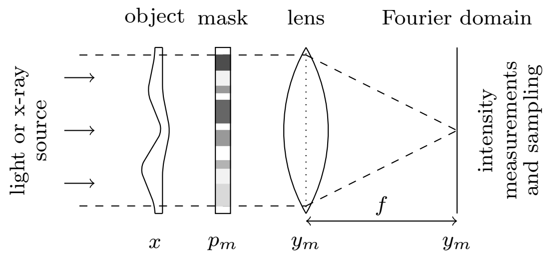


Figure 5.1.: A typical setup of a phase retrieval problem in optics.

5.2. Measurement by structured modulation

We follow the measurement setup introduced in e.g. [60], which is inspired by the method in optics. The corresponding setup is sketched in Fig. 5.1 and the corresponding signal flow diagram is depicted in Fig. 5.2.

In this measurement setup, the object, which one aims to investigate, is illuminated by coherent light source. By this process, one obtains a diffraction pattern, which contains needed informations about the object, which appears in the far-field. The occuring diffraction pattern corresponds to the Fourier transform of our signal of interests. If the light source is a light beam, and the object is a crystal, the inverse Fourier transform of the diffraction pattern contains the information about the electron density.

For convenient, we consider especially square-integrable signal (call h) which is supported on the rectangle $E := [-T_1/2, T_1/2] \times [-T_2/2, T_2/2]$, with $T_1, T_2 > 0$ a real constant, i.e. $h \in L^2(Q_{T_1,T_2})$. We shall see later, that by some simple modifications, the algorithm introduced here can also be applied to signals which is supported on other appropriate subsets of $\mathbb{R}^N$. Beforehand, we mention that this subset can not be of arbitrary shape.

We add for each M measurement steps a spatial light modulator directly behind the object, resulting the modified signals $y^{(m)}, m \in [M]$. Each of M spatial light modulator has the form:

$$p^{(m)}(t) = \sum_{k=1}^{K} \overline{\alpha_k^{(m)}} e^{-i\langle \lambda_k, x\rangle}, \quad t \in \mathbb{R}^2,\ m \in [M],$$

for some constants $\alpha_k^{(m)} \in \mathbb{C}$, $k \in [K]$, and some collection $\{\lambda_k\}_k$ of vectors in $\mathbb{R}^2$. Both shall be fixed soon. The resulted modified signal $y^{(m)}, m \in [M]$ are each simply the multiplication of h with $p^{(m)}$. The lens conduces to project the far field to close range. The signals which reaches the detector is, as we already mention, basically the Fourier transform of $y^{(m)}$, $m \in [M]$. By straightforward calculation, one gets the following expression of the latter:

$$\hat{y}^{(m)}(t) = \sum_{k=1}^{K} \overline{\alpha_k^{(m)}} \hat{h}(z + \lambda_k). \tag{5.1}$$

As we already know, the detector can only record the squared modulus of each signal $\hat{y}^{(m)}$. By Two-dimensional uniform sampling the signal recorded by the detector by some rate $1/\beta_1\beta_2$, $\beta_1, \beta_2 > 0$, we obtain for each $m \in [M]$ and $n \in \mathbb{Z}^2$ the following:

$$c_n^{(m)} = |\hat{y}^{(m)}(n \odot \beta)|^2 = |\sum_{k=1}^{k=K} \overline{\alpha_k^{(m)}} \hat{h}(n \odot \beta + \lambda_k)|^2,$$

where $\odot$ denotes the pointwise multiplication: $n \odot \beta = (n_1\beta_1, n_2\beta_2)$. The corresponding rate and respectively the uniform sampling pattern shall be fixed soon. For each $m \in [M]$ and $n \in \mathbb{Z}^2$, to give an

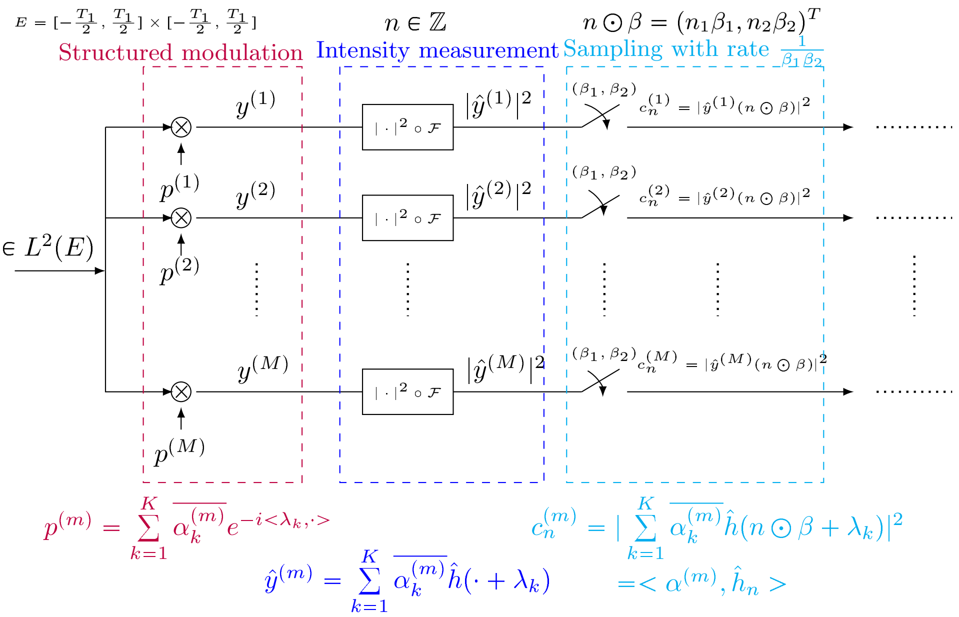


Figure 5.2.: Signal flow description of the measurement by structured modulation method.

alternative description of $c_n^{(m)}$, we define the vectors:

$$\boldsymbol{\alpha}^{(m)} := \begin{pmatrix} \alpha_1^{(m)} \\ \vdots \\ \alpha_K^{(M)} \end{pmatrix}, \quad \text{and} \quad \hat{\boldsymbol{h}}_n := \begin{pmatrix} \hat{h}(\lambda_1^{(n)}) \\ \vdots \\ \hat{h}(\lambda_K^{(n)}) \end{pmatrix} := \begin{pmatrix} \hat{h}(n \odot \beta + \lambda_1) \\ \vdots \\ \hat{h}(n \odot \beta + \lambda_K) \end{pmatrix}.$$

Accordingly, we have the following alternative description of $c_n^{(m)}$

$$c_n^{(m)} = |\langle \hat{\mathbf{h}}_n, \boldsymbol{\alpha}^{(m)} \rangle|^2, \quad n \in \mathbb{Z}^2,$$

By this reformulation of $c_n^{(m)}$, it stands clear, that for each measurement $n \in \mathbb{Z}^2$, we can apply finite-dimensional phase retrieval given in Thm. 5.1 to reconstruct $\hat{\boldsymbol{h}}_n$ from $\{c_n^{(m)}\}_m$, provided that $\{\boldsymbol{\alpha}^{(m)}\}_m$ forms a 2-uniform M/K-tight frame, which we now of course require. Specifically, one can compute for each $n \in \mathbb{Z}^2$ the following matrix:

$$Q_{\hat{\boldsymbol{h}}_n} = \frac{K(K+1)}{M} \sum_{m=1}^{M} c^{(m)} Q_{a^{(m)}} - c^{(m)} \,\mathrm{Id}\,. \tag{5.2}$$

Hence, for each $n \in \mathbb{Z}^2$ we get a block of measurements $\hat{\boldsymbol{h}}_n$ from $\{c_n^{(m)}\}_m$ multiplied by a constant phase θ_n, i.e. $\hat{\boldsymbol{h}}_n e^{i\theta_n}$, which resulted from the decomposition of the matrix $Q_{\hat{\boldsymbol{h}}_n}$ into $\hat{\boldsymbol{h}}_n e^{i\theta_n}(\hat{\boldsymbol{h}}_n e^{i\theta_n})^*$. Generally, $\{\theta_n\}_{n \in \mathbb{Z}^2}$ mutually differs. By a slight modification of the previous mentioned method, and by phase propagation analogue to the 1 variable case as described in [60, 48], we can determine each $\{\theta_n\}$ s.t. they are equally. In the following, we introduce the corresponding method.

Let $n \in \mathbb{Z}^2$ be arbitrary. We already know that by formula (5.2), the rank-1 matrix $Q_{\hat{\boldsymbol{h}}_n}$ can be determined. Further, suppose that we know the phase $\phi_{n,j}$ of an entry of $\hat{\boldsymbol{h}}_n$, i.e. $\phi_{n,j} = \arg([\hat{\boldsymbol{h}}_n]_i)$, for a $i \in [M]$. Then the whole vector $\hat{\boldsymbol{h}}_n$ can be determined by:

$$\hat{h}(n \odot \beta + \lambda_k) = \sqrt{[Q_{\hat{\boldsymbol{h}}_n}]_{k,k}} e^{i(\phi_{n,j}) - \arg([Q_{\hat{\boldsymbol{h}}_n}]_{j,k})}, \quad k \neq j. \tag{5.3}$$

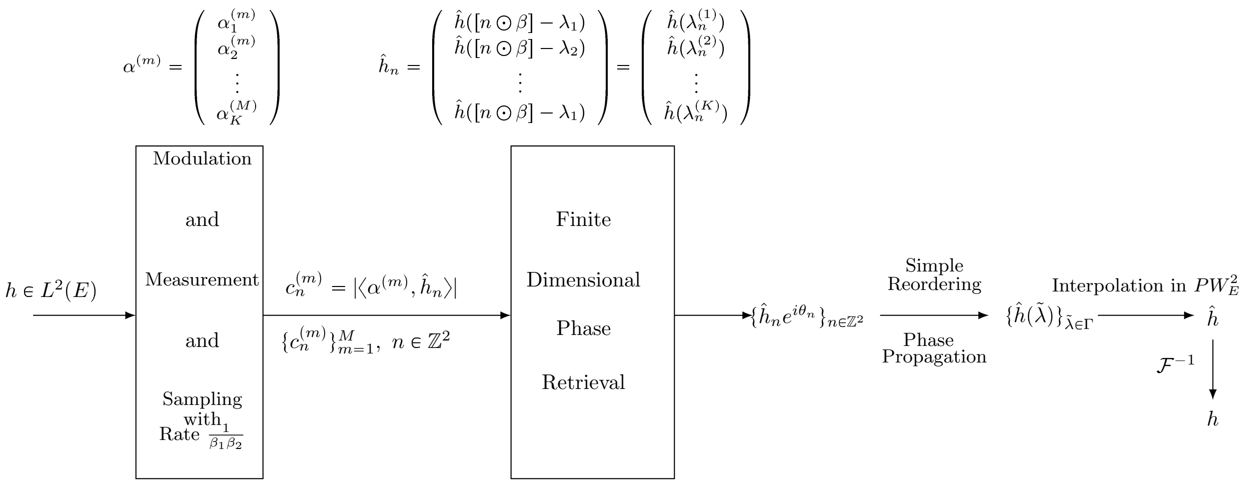


Figure 5.3.: Recovery of h from the intensity measurements

Now, for each $n \in \mathbb{Z}^2$, define the set Λ_n as the set, which contains the entries of $\hat{\boldsymbol{h}}_n$ (So, it stands clear to see Λ_n also as a block of measurements). Now, for each $n = (n_1, n_2) \in \mathbb{Z}^2$, we define another set related to the collection of blocks $\{\Lambda_n\}_n$:

$$\Lambda^{(1)}_{(n_1,n_2)} := \Lambda_{(n_1,n_2)} \cap \Lambda_{(n_1+1,k_2)}, \quad \text{and} \quad \Lambda^{(2)}_{(n_1,n_2)} := \Lambda_{(n_1,n_2)} \cap \Lambda_{(n_1,k_2+1)}. \tag{5.4}$$

To say in a sloppy way, $\Lambda^{(1)}_{(n_1,n_2)}$ and $\Lambda^{(2)}_{(n_1,n_2)}$ are sets, which contain the overlapping points of subsequent blocks. To ensure that $\Lambda^{(1)}_{(n_1,n_2)}$ and $\Lambda^{(2)}_{(n_1,n_2)}$ are non-empty, we require that for all $n = (n_1, n_2) \in \mathbb{Z}^2$, the following holds:

$$\Lambda_{(n_1,n_2)} \cap \Lambda_{(k_1,k_2)} \neq \varnothing,\ \ k_1 \in \{n_1, n_1+1\}, k_2 \in \{n_2, n_2+1\}, k_1 \neq k_2. \tag{5.5}$$

Furthermore, we require that $\Lambda^{(1)}_{(n_1,n_2)}$ and $\Lambda^{(2)}_{(n_1,n_2)}$ contain a non-zero element (we shall see the reason for this requirement soon). Now, for an "initial" block Λ_n, $n \in \mathbb{Z}^2$, assume that we know $\Lambda_{(n_1,n_2)}$. Now we aim to find out $\Lambda_{(n_1+1,n_2)}$, s.t. its phase $\phi_{(n_1+1,n_2)}$ coincide with ϕ_n (the subsequent approach should also be applied to the block $\Lambda_{(n_1,n_2+1)}$, respectively). Suppose that we have already find out $\Lambda_{(n_1,n_2)}$ by the knowledge of the "real" phase of an element of this set and by applying (5.3). Take a non-zero[1] element from the non-empty set $\Lambda^{(1)}_{(n_1,n_2)}$, which is $[\hat{\boldsymbol{h}}_(n_1, n_2)]_j$, for a $j \in [K]$. Accordingly, we have phase knowledge of a non-zero entry of $\hat{\boldsymbol{h}}_(n_1, n_2)$. So, we can reconstruct $\hat{\boldsymbol{h}}_(n_1, n_2)$ by (5.3). So starting from a certain "initial" block, where the knowledge of the phase is available, one may apply above method until all the desired points (which is denoted later by Γ) are obtained.

Next, we unite and simply reorder the samples which resulted from previous processes. Accordingly we obtain the samples $\{\hat{h}(\lambda)\}_{\lambda \in \Gamma}$ up to an overall constant phase θ, where Γ is defined as the overall sampling points:

$$\Gamma := \bigcup_{n \in \mathbb{Z}^2} \boldsymbol{\lambda}_n,$$

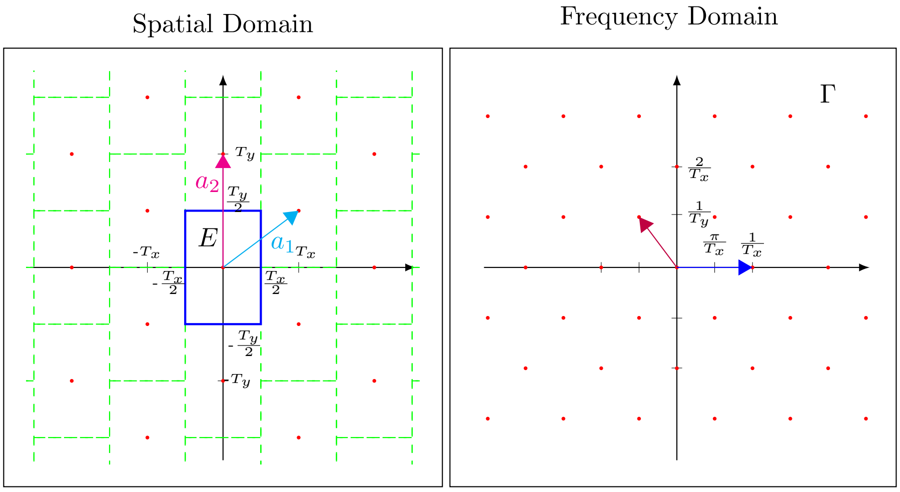
where $\boldsymbol{\lambda}_n = \{\lambda_m^{(n)}\}_{m=1}^K$.

Since $h \in L^2(E)$ and accordingly $\hat{h} \in \tilde{\mathcal{PW}}_E^2$, we can interpolate $\hat{h}$ from its samples taken in the points Γ by corollary 4.28, provided that Γ is a lattice, whose (almost) fundamental domain is E. Finally, inverse Fourier transforming $\hat{h}$ yields h up to a constant phase. A summary of the algorithm, given in this section, is given in fig. 5.2

[1]Notice that otherwise this method would no be applicable

5.3. Example

First, we choose the sampling points as follows:



E tiles $\mathbb{R}^2$ by translation set Λ
$\Lambda = A\mathbb{Z}^2, \quad A = [a_1, a_2]$
$a_1 = [T_x, \frac{T_y}{2}] \qquad a_2 = [0, T_y]$

$\Lambda^* =: \Gamma$ spectrum of $L^2(E)$
$\Lambda^* = B\mathbb{Z}^2, \; B = [b_1, b_2], \; B = A^{-T}$
$b_1 = [\frac{1}{T_x}, 0]^T \qquad b_2 = [-\frac{1}{2T_x}, \frac{1}{T_y}]^T$

Figure 5.4.: Constructing a spectrum for E

We copy the area E, wherein the signal h is supported on, and shift the copies of E by translation set $\Lambda := A\mathbb{Z}^2$, where

$$A := \begin{pmatrix} T_x & 0 \\ \frac{T_y}{2} & T_y \end{pmatrix}.$$

Obviously, E tiles $\mathbb{R}^2$ by this way, or equivalently, E is an almost fundamental domain of Λ. Corollary 4.28 asserts that the dual lattice of Λ is the desired sampling points (the corresponding reconstruction formula can also be found in that corollary). We denote the dual lattice of Λ by Γ. By simple computation, we obtain $\Gamma = B\mathbb{Z}^2$, where

$$B = A^{-T} = \begin{pmatrix} \frac{1}{T_x} & -\frac{1}{2T_x} \\ 0 & \frac{1}{T_y} \end{pmatrix}.$$

Consequently, the interpolation formula of the fourier transform $\hat{h}$ of $h \in L^2(E)$, given in corr 4.28, which is in $\mathcal{PW}_E^2$ can be written explicitly as:

$$\begin{aligned} \hat{h}(\xi) \quad &= \sum_{n_1, n_2 \in \mathbb{Z}} \hat{h}(\frac{1}{T_x}n_1 - \frac{1}{2T_x}n_2, \frac{1}{T_y}) \\ &\cdot \quad \operatorname{sinc}(\frac{T_x}{2}[\xi_1 - (\frac{1}{T_x}n_1 - \frac{1}{2T_x}n_2)]) \cdot \operatorname{sinc}(\frac{T_y}{2}[\xi_2 - \frac{1}{T_y}n_2]), \end{aligned}$$

We choose the corresponding 4 modulation coefficients as follows:

$$\lambda_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 1/2T_x \\ 1/T_y \end{pmatrix}, \; \lambda_3 = \begin{pmatrix} 1/T_x \\ 0 \end{pmatrix}, \; \lambda_4 = \begin{pmatrix} 1/T_x \\ -1/T_y \end{pmatrix}.$$

As depicted above, we can see that subsequently blocks containing shifted versions of $\{\lambda_k\}_{k=1}^4$ have each one overlapping and the union of the blocks is exactly Γ. As we have seen in the main part, the signals which can be reconstructed are those, whose Fourier transform is non-zero at all of those overlapping points.

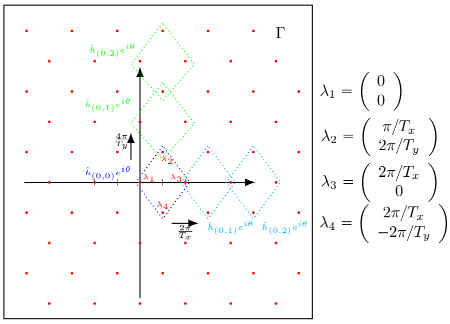


Figure 5.5.: Approximating Γ by $\{\boldsymbol{\lambda}_n\}_{n\in\mathbb{Z}^2}$

Now, we need to determine a 2-uniform M/K tight frame for $\mathbb{C}^4$. We may construct this particular generating system by the following way (see [62]):

- First, We define the 4×4 matrices U and V:

$$U := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{4}} & 0 & 0 \\ 0 & 0 & e^{\frac{4\pi i}{4}} & 0 \\ 0 & 0 & 0 & e^{\frac{6\pi i}{4}} \end{pmatrix}, \quad V := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- Let $\rho = e^{\frac{i\pi}{4}}$. We define the vectors ψ_{1a} and ψ_{1b}:

$$\psi_{1a} := \frac{1}{\sqrt{6}}\begin{pmatrix} \rho+1 \\ i \\ \rho-1 \\ i \end{pmatrix}, \quad \psi_{1b} := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

- Let $X := \frac{1}{2}\sqrt{3-\frac{3}{\sqrt{5}}}$ and $Y := \frac{1}{2}\sqrt{1+\frac{3}{\sqrt{5}}}$.We define ψ_k:

$$\psi_k = X\psi_{1a} + \rho^k Y\psi_{1b}, \quad k = 1,3,5,7$$

- We choose $k = 1,3,5,7$ arbitrary. We obtain the family of vectors $D_{4,k} := \{V^c U^d \psi_k : c,d \in \mathbb{Z}_4\}$, which is a maximal 2-uniform M/K-tight frame for $\mathbb{C}^4$

So, a possible choice of $\{\boldsymbol{\alpha}^{(m)}\}_{m=1}^4$ is $D_{4,k}$ with $k \in \{1,3,5,7\}$ chosen arbitrarily. By this special choice of interpolation points and mask, we obtain the overall sampling rate 8 times the 2-dimensional Nyquist-rate T_xT_y (Notice that the factor 2π is not existing here, by our choice of the Fourier transform), which is lower than the expected rate of 16 times the 2-dimensional Nyquist-rate. This can be seen if we consider the signal flow diagram of this specific sampling scheme: it contains 16 signal branches with each sample taken by the rate $\frac{T_xT_y}{2}$. The corresponding signal

6. A Class of Weighted Frames of Exponentials and Sampling of Multiband Signal

For convenient, we begin by introducing some notions and by giving some conventions used throughout this chapter. Given a quadratic matrix $\mathrm{H} \in \mathbb{C}^{N\times N}$. We denote by $\lambda_{\min}(\mathrm{H})$ (resp. $\lambda_{\max}(\mathrm{H})$) the smallest (resp. the biggest) eigenvalue of the matrix H. Let $\mathrm{H} \in \mathbb{C}^{M\times N}$, as similar, we define $\sigma_{\min}(\mathrm{H})$ (resp. $\sigma_{\max}(\mathrm{H})$) as the smallest (resp. biggest) singular value of H. Given a collection of subsets $\{D_n\}_{n\in\mathcal{I}}$ of a measureable space D. We say $\{D_n\}_{n\in\mathcal{I}}$ is a measureable partition of D, if each set in that collection is measureable, the sets are mutually disjoint, and their union is D. Let $x, y \in \mathbb{R}$, we denote $x \asymp y$, if there exists positive constants A, B, s.t. $Ay \leqslant x \leqslant By$. The first term given in the following definition constitutes the "bands" of the signal of our interests. Besides, we formalize also therein the term "periodization":

Definition 6.1 (K-Fundamental Domain, Periodization). *Let Λ be a uniform lattice in an LCA group. A subset $\mathbb{B}$ of $\mathcal{G}$ is said to be a K-fundamental domain of Λ, if $\mathbb{B}$ is a union of K mutually disjoint fundamental domains, i.e. $\mathbb{B} := \bigcup_{k\in[K]} \Omega_k$, where Ω_k is a fundamental domain for Λ, $\forall k \in [K]$, and $\Omega_k \cap \Omega_l = \varnothing$, if $k \neq l$.*

Let D be a fundamental domain of Λ, and $\mathbb{B}$ be a K-fundamental domain of Λ. For an $h \in L^2(D)$, we define the periodization $(\cdot)^{\mathbb{B}}$ of h w.r.t. the countable uniform lattice Λ to $\mathbb{B}$ by:

$$h^{\mathbb{B}} := \sum_{\lambda\in\Lambda} h(\cdot - \lambda). \tag{6.1}$$

Throughout this chapter, it always holds $N, M, K \in \mathbb{N}$, and $M \geqslant K$.

Let $\mathbb{B}$ be a K-fundamental domains of an admissible uniform lattice Λ in an LCA group $\mathcal{G}$. In this chapter, we concern ourselves with construction of a class of frames for the Hilbert space $L^2(\mathbb{B})$. More specific, let $\{\phi^{(m)}\}_{m\in[M]}$ be a collection of functions (whose restriction to $\mathbb{B}$ is) measureable on $\mathbb{B}$, and D be a fundamental domain of Λ. The frame for $L^2(\mathbb{B})$, which we aim to construct in this chapter, is the collection, consists of the functions $\phi^{(m)}\lambda$, $m \in [M]$, $\lambda \in \mathrm{A}_{\mathcal{G}^\wedge}(\Lambda)$.

For easiness in later approaches, we introduce the following notations:

Notations 1. *Let Λ be an admissible uniform lattice in an LCA group, and let $\tilde{\Lambda}$ be an admissible uniform lattice in $\mathcal{G}^\wedge$. We denote the annihilator of Λ in $\mathcal{G}^\wedge$ by $\Lambda^{\perp}_{\mathcal{G}^\wedge}$ instead of $\mathrm{A}_{\mathcal{G}^\wedge}$. In case that it is clear from context, we denote $\Lambda^{\perp}_{\mathcal{G}}$ simply by $\Lambda^{\perp}$. We denote the annihilator of $\tilde{\Lambda} \subseteq \mathcal{G}^\wedge$ in $\mathcal{G}^{\wedge\wedge}$ by $\tilde{\Lambda}^{\perp}_{\mathcal{G}^{\wedge\wedge}}$ instead of $\mathrm{A}_{\mathcal{G}^{\wedge\wedge}}(\Lambda)$, and the annihilator of $\tilde{\Lambda} \subseteq \mathcal{G}^\wedge$ in $\mathcal{G}$ by $\tilde{\Lambda}^{\perp}_{\mathcal{G}}$. Again, in case that it is clear from context, we write the annihilator of $\tilde{\Lambda}$ in $\mathcal{G}^{\wedge\wedge}$ or in $\mathcal{G}$ simply by $\tilde{\Lambda}^{\perp}$. Further, $\tilde{\Lambda}^{\perp}$ shall mostly be interpreted as $\tilde{\Lambda}^{\perp}_{\mathcal{G}}$.*

To emphasize the analogy to the Fourier Basis in Euclidean spaces, we write for an admissible uniform lattice $\Lambda \subseteq \mathcal{G}$: $e_\lambda := \lambda(\cdot), \quad \forall \lambda \in \Lambda^{\perp}$. Let D be a relatively compact fundamental domain of Λ. As we already know, $\{e_\lambda\}_\lambda$ forms an ONB for $L^2(\mathcal{G}/\Lambda)$, and $\{e_\lambda \chi_D\}_\lambda$ forms an ONB for $L^2(D)$, where D is a relatively compact fundamental domain of Λ (Of course, the preceeding statements are true if conventions 4.12 are followed, which we do[1] throughout this chapter, if not otherwise stated) Sometimes, we shall slackly write $\{e_\lambda\}_\lambda$ instead of $\{e_\lambda \chi_D\}_\lambda$.

[1]This shall be mention explicitly soon.

Let $\tilde{\Lambda}$ be an admissible uniform lattice in $\mathcal{G}^\wedge$, and $\tilde{D}$ a relatively compact fundamental domain of Λ. We write $e_\lambda := \alpha_{\mathcal{G}}(\lambda)(\cdot), \quad \forall \lambda \in \tilde{\Lambda}^\perp_{\mathcal{G}}$. Clearly $\{e_\lambda\}_\lambda$ forms an ONB for $L^2(\mathcal{G}^\wedge/\tilde{\Lambda})$, and $\{e_\lambda \chi_{\tilde{D}}\}_\lambda$ is an ONB for $L^2(\tilde{D})$, for a relatively compact fundamental domain D of Λ, if $\mu_{\mathcal{G}^\wedge}$, $\mu_{\tilde{\Lambda}}$, $\mu_{\mathcal{G}^\wedge/\tilde{\Lambda}}$, and the measure on $\tilde{D}$ are chosen according to conventions 4.12. Correspondingly, the Haar measure on $\mathcal{G}$ is chosen s.t. the Fourier inversion formula holds. If there is no danger for confusions, we write $\{e_\lambda\}_\lambda$ instead of $\{e_\lambda \chi_{\tilde{D}}\}_\lambda$.

Choose a "nice"[2] fundamental domain D of Λ. In this chapter, it shall particularly be shown, that the condition on the collection of functions $\{\phi^{(m)}\}_{m\in M}$, s.t. $\{\phi^{(m)} e_\lambda : m \in [M], \lambda \in \Lambda^\perp\}$ forms a frame for $L^2(\mathbb{B})$, is closely related to the following matrix which varies on a chosen fundamental domain D of Λ:

$$\Phi(x) = \begin{pmatrix} \phi^{(1)}(x+\lambda_1(x)) & \dots & \phi^{(M)}(x+\lambda_1(x)) \\ \vdots & \ddots & \vdots \\ \phi^{(1)}(x+\lambda_K(x)) & \dots & \phi^{(M)}(x+\lambda_K(x)) \end{pmatrix}, \quad x \in D, \tag{6.2}$$

where each $\lambda_k(\cdot)$, $k \in [K]$, is a mapping from D to Λ. We further call Φ simply *the varying matrix formed by* $\{\phi^{(m)}\}_{m\in M}$. Notice, that each $\lambda_k(\cdot)$, $k \in [K]$ can vary arbitrarily on each points of D, which may complicate the analysis on G. Furthermore, for measure theoretical reason, we have to require that $\Phi(\cdot)$ is measureable, in the sense, that each functions in its entry is measureable. Explicitly, let $\Omega \subseteq \mathcal{G}$, and $p, q \in \mathbb{N}$, and given a varying matrix $\mathrm{H} : \Omega \to \mathbb{C}^{p\times q}$, which has the form:

$$\mathrm{H}(\cdot) = \begin{pmatrix} h^{(1,1)}(\cdot) & \dots & h^{(1,q)}((\cdot)) \\ \vdots & \ddots & \vdots \\ h^{(p,1)}(\cdot) & \dots & h^{(p,q)}(\cdot) \end{pmatrix},$$

where $h^{(k,l)}$, for each $k \in [p]$ and $l \in [q]$, is a function $h^{(k,l)} : \Omega \to \mathbb{C}$. We say H is measureable, if for each $k \in [p]$ and $l \in [q]$, $h^{(k,l)} : \Omega \to \mathbb{C}$ is measureable. To emphasize the fact that Φ is generated by $\{\phi^{(m)}\}_{m\in[M]}$, we write simply Φ_ϕ.

Now, we get back to our discussion. Once we have specified the collection of mapping $\{\lambda_k(\cdot)\}_{k\in[K]}$, and required $\mathbb{B}$ and D to be relatively compact, we shall later see that each $\lambda_k(\cdot)$ basically varies only finitely times on D. To avoid repetitions, we commit ourselves to the following notations:

Notations 2. *If not otherwise stated, Λ stands for an admissible uniform lattice in an LCA group $\mathcal{G}$. To Λ, we assign a "basis" fundamental domain D, and a K-fundamental domain $\mathbb{B}$ (which is the domain of the function space, for which we want to find a frame). The Haar measures on $\mathcal{G}$, Λ, and D are chosen by conventions 4.12, and the measure on $\mathbb{B}$ is simply the subspace measure inherited from $\mathcal{G}$.*

When deriving sampling formula for multi-banded functions, the following notations shall be used: As similar, $\tilde{\Lambda}$ stands for an admissible uniform lattice in the dual group $\mathcal{G}^\wedge$ of an LCA group $\mathcal{G}$. To $\tilde{\Lambda}$, we assign a "basis" fundamental domain $\tilde{D}$, and a K-fundamental domain $\tilde{\mathbb{B}}$. When we consider this case, we shall choose the Haar measures on $\mathcal{G}^\wedge$, $\tilde{\Lambda}$, $\tilde{D}$ by conventions 4.12. Furthermore the measure on $\tilde{\mathbb{B}}$ is chosen as subspace measure inherited from $\mathcal{G}^\wedge$. The measure on $\mathcal{G}$ is chosen as usual s.t. the Fourier inversion formula holds.

As an analogy to the well-known Euclidean case, we call a square-integrable signal f multi-banded, if its Fourier transform is supported on the union of subsets of its frequency domain. In this work, we restrict ourselves to multi-banded signals, whose bands are contained in a finite union of relatively compact fundamental domains of some admissible lattices in $\mathcal{G}^\wedge$. Now, let $\tilde{\Lambda}$ be an admissible uniform lattice in $\mathcal{G}^\wedge$, and given a collection of functions $\{g^{(m)}\}_{m\in[M]}$ each in $L^2(\tilde{\mathbb{B}})$. Suppose that $\{g^{(m)} e_\lambda : m \in [M], \lambda \in \tilde{\Lambda}^\perp_{\mathcal{G}}\}$[3]

[2] As we shall see later, the fundamental domain D constitutes a "basis" domain for our analysis. So, it stands clear to choose a "primitive" D, e.g. in case $\mathcal{G} = \mathbb{R}^N$, a fundamental parallepiped of Λ is a good choice for D

[3] we take the annihilator of Λ in $\mathcal{G}$, since the annihilator of Λ in $\mathcal{G}^{\wedge\wedge}$ can be identified with previously mentioned by Pontryagin-van Kampen duality

forms a frame for $L^2(\tilde{\mathbb{B}})$. The frame[4] $\{g^{(m)}e_{-\lambda}\}$ for $L^2(\tilde{\mathbb{B}})$ is closely connected to the sampling process of multi-banded functions on LCA groups, as we shall see in the following discussion:

For a multi-banded signal $f \in \mathcal{PW}^2_{\tilde{\mathbb{B}}}$, One can easily compute the coefficients of the frame expansion of its spectrum $\hat{f} \in L^2(\tilde{\mathbb{B}})$, w.r.t. $\{g^{(m)}e_{-\lambda}\}$ as follows:

$$\begin{aligned}
c_\lambda^{(m)} &= \langle \hat{f}, g^{(m)}e_{-\lambda}\rangle_{L^2(\tilde{\mathbb{B}})} = \int_{\tilde{\mathbb{B}}} \hat{f}(\gamma)\overline{g^{(m)}}(\gamma)\alpha_{\mathcal{G}}(\lambda)\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma) \\
&= \int_{\mathcal{G}^\wedge} \hat{f}(\gamma)\overline{g^{(m)}(\gamma)}\alpha_{\mathcal{G}}(\lambda)\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma) = \int_{\mathcal{G}^\wedge} \hat{f}(\gamma)\overline{g^{(m)}}\ \overline{\gamma(\lambda)}\mathrm{d}\mu_{\mathcal{G}^\wedge}(\gamma) \\
&= (f * \overline{\check{g}^{(m)}})(\lambda), \quad \text{for } m \in [M],\ \lambda \in \Lambda_{\mathcal{G}}^{\perp},
\end{aligned} \tag{6.3}$$

where the third equality follows from the fact that $\hat{f}$ is supported in $\tilde{\mathbb{B}}$. Let[5] $\{\tilde{\phi}_{m,\lambda} : m \in [M],\ \lambda \in \tilde{\Lambda}^{\perp}\}$ be a dual frame[6] (e.g. the canonical dual frame) of $\{g^{(m)}e_{-\lambda}\}_{m,\lambda}$. It is clear, that $\hat{f}$ can be expanded as follows:

$$\hat{f} = \sum_{\substack{m\in[M]\\ \lambda\in\tilde{\Lambda}^{\perp}}} c_\lambda^{(m)}\phi_{m,\lambda} = \sum_{m,\lambda}(f * \overline{\check{g}^{(m)}})(\lambda)\tilde{\phi}_{m,\lambda}.$$

Finally by inverse Fourier transforming[7] above expression, one obtains the following expansion:

$$f = \sum_{\substack{m\in[M]\\ \lambda\in\tilde{\Lambda}^{\perp}}} (f * \overline{\check{g}^{(m)}})(\lambda)\ \mathcal{F}^{-1}\{\chi_{\tilde{\mathbb{B}}}\tilde{\phi}_{m,\lambda}\},$$

where the convergence is in L^2-sense, and correspondingly, the equality in a.e.-sense. Besides, it will be shown, that the function, which is resulted by this process is continuous and vanish at infinity.

So, above computations, proposes a sampling and reconstruction scheme for multi-banded multi-dimensional square integrable function (for illustration see fig. 6.1):

- Obtain the samples $\{c_\lambda^{(m)}\}_{\lambda\in\tilde{\Lambda}^{\perp}}$ for each $m \in [M]$ by filtering the signal f by the linear filter with the frequency response $\overline{g^{(m)}}$, and subsequently, sample the resulted signal on the points $\tilde{\Lambda}^{\perp}$ in $\mathcal{G}$. (see fig. 6.1).
- Reconstruct the signal f from the samples of its preprocessed version $\{c_\lambda^{(m)}\}_{m,\lambda}$, by the formula $f = \sum_{n,\lambda} c_\lambda^{(m)}\rho_{m,\lambda}$, where $\rho_{m,\lambda} := \mathcal{F}^{-1}\{\tilde{\phi}_{m,\lambda}\}$, for each $m \in [M]$, $\lambda \in \tilde{\Lambda}^{\perp}$.

Before we give the desired sufficient and necesary conditions on the collection of functions $\{\phi^{(m)}\}_m$, s.t. $\{\phi^{(m)}e_\lambda\}$ forms a frame for $L^2(\mathbb{B})$, we first give some auxiliary statements in the following section:

6.1. Auxiliary Statements

The following lemma constitutes a step to establish an isometric isomorphism between $L^2(D)^{\oplus K}$ and $L^2(\mathbb{B})$. Notice that, as $\mathbb{B}$ is a K-fundamental domain of a countable uniform lattice Λ in $\mathcal{G}$, $\mathbb{B}$ is the union of a mutually disjoint collection $\{\Omega_k\}_{k\in[K]}$ of fundamental domains for Λ. Specifically in the following lemma, we want to part the "reference" fundamental domain D possibly into finite collection $\{D_n\}_{n\in\mathcal{I}}$ of subsets, s.t. each Ω_k, $k \in [K]$, is basically the union of the translated versions of each $\{D_n\}_{n\in\mathcal{I}}$. This construction is possible, since, as we have seen, fundamental domains constitute lattice equivalent sets.

[4] Notice that a frame is an unconditional basis, so the permutation of its elements yields also a frame

[5] we write $\phi_{m,\lambda}$ instead of $\phi_\lambda^{(m)}$, since we do not know whether $\phi_{m,\lambda}$ can be written as $h^{(m)}e_\lambda$, for a function $h^{(m)}$ measureable on $\tilde{\mathbb{B}}$.

[6] In some cases, it is not easy to compute the dual frame of a frame

[7] Notice that linear homeomorphisms preserve frame, inverse Fourier transformation is a linear homeomorphism, and $\mathcal{PW}_\Omega$ is a closed subspace of $L^2(\mathcal{G})$

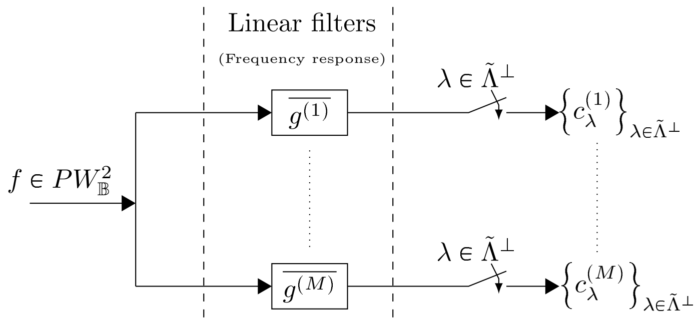


Figure 6.1.: Sampling device of multi-band multi-dimensional square-integrable signal induced by the frame $\{g^{(m)}e_\lambda\}_{m,\lambda}$ for $L^2(\mathbb{B})$

Lemma 6.1. *Let D be a relatively compact fundamental domain of a countable uniform lattice Λ in an LCA group $\mathcal{G}$. Given a finite collection $\{\Omega_k\}_{k\in[K]}$ of relatively compact fundamental domain of Λ. Then there exists a finite measureable partition $\{D_n\}_{n\in\mathcal{I}}$ of D, s.t. for each $k \in [K]$ there exists a finite measureable partition $\{\Omega_k^{(n)}\}_{n\in\mathcal{I}}$ of Ω_k and a finite sequence $\{\lambda_n^{(k)}\}_{n\in\mathcal{I}}$ of Λ fulfilling:*

$$D_n + \lambda_n^{(j)} = \Omega_j^{(n)}, \quad \forall n \in \mathcal{I}.$$

Proof. For each $k \in [K]$, part D into mutually disjoint measureable subsets w.r.t. Ω_k as suggested by lemma 4.7 into finite collection of mutually disjoint measureable subsets $\{D_n^{(k)}\}_{n\in\mathcal{I}^{(k)}}$, and associate to each member $D_n^{(k)}$ of this partition the corresponding lattice element $\lambda_n^{(k)} \in \Lambda$.

For each $k \in [K]$, define the set of tuples $\mathcal{J}^{(k)} := \{(D_n^{(k)}, \lambda_n^{(k)})\}_{n\in\mathcal{I}^{(k)}}$ to emphasize the connection between $D_n^{(k)}$ and $\lambda_n^{(k)}$.

Now, we define another the set of tuples $\mathcal{J}$ as follows:

$$\mathcal{J} := \left\{ (X,\Gamma) : X = \bigcap_{k\in[K]} D^{(k)},\ \Gamma = \{\lambda^{(k)}\}_{k\in[K]},\ \text{where } (D^{(k)}, \lambda^{(k)}) \in \mathcal{J}^{(k)},\ \forall k \in [K] \right\}.$$

In other words, an element $(X, \{\lambda^{(k)}\}_{k\in[K]})$ of $\mathcal{J}$ consists of a subset X of D, and the corresponding lattice sequence $\{\lambda^{(k)}\}_{k\in[K]}$, for which $X + \lambda^{(k)} \subseteq \Omega_k$. Renumerate (the index k of each $\lambda^{(k)}$ over $[K]$ remains unchanged) $\mathcal{J}$ as follows: $\mathcal{J} =: \{(D_n, \{\lambda_n^{(k)}\}_{k\in[K]})\}_{n\in\mathcal{I}}$. It is not hard to see that $|\mathcal{I}| \leqslant 2^{\sum_{k\in[K]}|\mathcal{I}^{(K)}|}$. Hence the partition $\{D_n\}_{n\in\mathcal{I}}$ of D is finite. Clearly, each D_n in $\{D_n\}_{n\in\mathcal{I}}$ is measureable, since each of them resulted from finite intersection of measureable sets. By more detailed observations on the definition of $\mathcal{J}$, one immediately obtains the statement. ∎

We call the partition of D given in above lemma as *the partition of* D w.r.t. $\{\Omega_k\}_k$. For later approaches, it is advantageous to call this partition *the partition of* D w.r.t. $\mathbb{B}$, where $\mathbb{B} = \bigcup_k \Omega_k$. Given a fundamental domain D of a countable lattice Λ, and K-collection of mutually disjoint fundamental domains $\{\Omega_k\}_{k\in[K]}$ of Λ (whose union is clearly a K-fundamental domain of Λ). By means of lemma 6.1, one is able to reduce the behaviour of each cross-transversals mappings $\{\tau_{D\to\Omega_k}\}_{k\in[K]}$ to a "common denominator", in the sense that: on D, particularly on each D_n, $n \in \tilde{I}$, each $\tau_{D\to\Omega_k}$, $k \in [K]$ can be written sectionwise as $p_k(\cdot) = (\cdot) + \lambda_n^{(k)}$.

To make the construction given in the proof of lemma 6.1 more comprehensible, we give in the following an elementary example:

Examples 6.2. *Consider the additive LCA group $\mathbb{R}$. Let the uniform lattice $\Lambda = \mathbb{Z}$ be given. Let be $D = [0,1)$, $\Omega_1 = [1.34, 2.34)$, and $\Omega_2 = [6.72, 7.72)$ be given. Clearly, D, Ω_1, and Ω_2 are fundamental* *domains of Λ. The corresponding the collection tuples $\mathcal{J}^k$, $k \in \{1, 2\}$, consisting of a partition of D and a subset of Λ, is given as follows:*

- $\mathcal{J}^1 = \{([0, 0.34), 2), ([0.34, 1), 1)\}$,
- $\mathcal{J}^2 = \{([0, 0.72), 7), ([0.72, 1), 6)\}$.

Hence the collection of tuples $\mathcal{J}$ consists of:

$$\mathcal{J} = \{([0, 0.34), \{2, 7\}), ([0.34, 0.72), \{1, 7\}), ([0.72, 1), \{1, 6\})\}.$$

The following easy lemma gives a connection between the norm of an operator, whose is simply a pointwise matrix-vector multiplication of its input with a varying matrix $\mathrm{G}(\cdot)$, and the norm of the gramian matrix corresponding to G:

Lemma 6.3. *Let $D \subset \mathcal{G}$ be a measureable subset of an LCA group. Given a mapping[8] $\mathrm{H} : D \to \mathbb{C}^{K\times M}$ measureable[9] on D. Suppose that there exist a countable measureable partition $\{D_n\}_{n\in\mathcal{I}}$ of D containing subsets of positive measure. Let $\mathrm{V} : L^2(D)^{\oplus M} \to L^2(D)^{\oplus K}$ be a mapping which is given by $x \mapsto \mathrm{H}x$. Then the following statements hold:*

(a) V *is bounded and surjective if and only if there exist constants $A, B > 0$ s.t. $\forall n \in \mathcal{I}$:*

$$A\,Id \leqslant \mathrm{H}(x)\mathrm{H}(x)^* \leqslant B\,Id, \quad \text{a.e. } x \in D_n. \tag{6.4}$$

(b) Furthermore, let $K = M$. V *is bounded and bijective if and only if there exist constants $A, B > 0$ s.t. $\forall n \in \mathcal{I}$:*

$$A\,Id \leqslant \mathrm{H}(x)^*\mathrm{H}(x) \leqslant B\,Id, \quad \text{a.e. } x \in D_n. \tag{6.5}$$

Proof. The linearity of V is obvious. One can easily convince oneself that the adjoint of V is given formally by $\mathrm{V}^* : L^2(D)^{\oplus K} \to L^2(D)^{\oplus M}$, $\mathbf{f} \mapsto \mathrm{H}^*\mathbf{f}$. Clearly, to show that V is a bounded surjective operator it is sufficient and necessary to show that $\|\mathrm{V}^*\mathbf{f}\| \asymp \|\mathbf{f}\|$, $\forall \mathbf{f} \in L^2(D)^{\oplus K}$. Compute the norm of the image of an $\mathbf{f} \in L^2(D)^{\oplus K}$ under V^* as follows:

$$\begin{aligned}\|\mathrm{V}^*\mathbf{f}\|^2 &= \int\limits_D \langle \mathrm{H}(x)\mathrm{H}(x)^*\mathbf{f}(x), \mathbf{f}(x)\rangle_{\mathbb{C}^K}\ \mathrm{d}\mu_{\mathcal{G}}(x) \\ &= \sum_{n\in\mathcal{I}} \int\limits_{D_n} \langle \mathrm{H}(x)\mathrm{H}(x)^*\mathbf{f}(x), \mathbf{f}(x)\rangle_{\mathbb{C}^K}\ \mathrm{d}\mu_{\mathcal{G}}(x).\end{aligned} \tag{6.6}$$

The second equality follows from the fact that $\{D_n\}_{n\in\mathcal{I}}$ is mutually disjoint, and its union is D. Notice, that $\langle \mathrm{H}(x)\mathrm{H}(x)^*\mathbf{f}(\omega), \mathbf{f}(x)\rangle$ is measureable[10] and that each D_n are assumed to be measureable.

For the left implication of (a), assume that (6.4) holds. It is not hard to see[11]that by this assumption, $\|\mathrm{V}^*\mathbf{f}\|^2 \asymp \|\mathbf{f}\|^2$.

For the right implication of (a): assume first that the right inequality in (6.4) does not hold, i.e. there exist a measureable subset $B \subseteq D$ with positive measure and an $\tilde{\mathbf{f}} \in L^2(D)^{\oplus K}$ s.t. $\langle \mathrm{H}(x)\mathrm{H}(x)^*\tilde{\mathbf{f}}(x), \tilde{\mathbf{f}}(x)\rangle = \infty$, $\forall x \in B$. Since B is measureable and B is contained in some D_n (the part of B in those D_n is clearly measureable), it follows immediately that $\int_{D_n} \langle \mathrm{H}(x)\mathrm{H}(x)^*\tilde{\mathbf{f}}(x), \tilde{\mathbf{f}}(x)\rangle \mathrm{d}\omega = \infty$ on those D_n. Hence, also

[8]One can imagine this as a $K \times M$ complex matrix which vary over D.

[9]i.e. each entry of this matrix is specified by a measureable function on D

[10]Notice that $\langle \mathrm{H}(x)\mathrm{H}(x)^*\mathbf{f}(x), \mathbf{f}(x)\rangle$ is principally sum of functions, which are each multiplication of measureable functions, and hence measureable.

[11]According to measure theory: Let B be a measure space with measure μ. For $\mathbf{f}, \tilde{\mathbf{f}} : B \to \overline{\mathbb{R}}$ quasiintegrable (i.e. the integral of the real part logical-or of the imaginary part of $\mathbf{f}$ (resp. $\tilde{\mathbf{f}}$) over B is bounded), B measureable subset of $\mathbb{R}^N$, and $\mathbf{f} \leqslant \tilde{\mathbf{f}}$ almost everywhere on D, it follows that $\int_B \mathbf{f}\mathrm{d}\mu \leqslant \int_B \tilde{\mathbf{f}}\mathrm{d}\mu$.

$\|\mathrm{V}^*\tilde{\mathbf{f}}\|^2 = \infty$. Now suppose that the left inequality in (6.4) does not hold, i.e. there exist a measureable subset $\tilde{U} \subseteq D$ with positive measure, and a function $\mathbf{f}' \in L^2(D)^{\oplus K}$ not identical zero almost everywhere on $\tilde{U}$ s.t. $\langle \mathrm{H}(x)\mathrm{H}(x)^*\mathbf{f}'(x), \mathbf{f}'(x)\rangle = 0$, $\forall \omega \in \tilde{U}$. Now, consider the trivial extension $\tilde{\mathbf{f}}$ of $\mathbf{f}'\big|_{\tilde{U}}$ to D. Clearly, it holds that $\tilde{\mathbf{f}} \in L^2(D)^{\oplus K}$, and finally it follows that $\|\mathrm{V}^*\tilde{\mathbf{f}}\| = 0$. So summarized, one can reason that $\|\mathrm{V}^*\mathbf{f}\|^2 \asymp \|\mathbf{f}\|^2$ if and only if:

$$\langle \mathrm{H}(x)\mathrm{H}(x)^*\mathbf{f}(x), \mathbf{f}(x)\rangle \asymp \langle \mathbf{f}(x), \mathbf{f}(x)\rangle, \ \forall \text{ a.e. } x \in D_n, \ \forall n \in \mathcal{I},$$

which corresponds to (a).

Now let $M = K$. Clearly V is bounded and bijective if and only if $\|\mathrm{V}\mathbf{f}\| \asymp \|\mathbf{f}\|$, $\forall \mathbf{f} \in L^2(D)^{\oplus K}$. By computations and argumentations similar to the first part of the proof, one obtains the second statement.∎

Remark 6.4. *Consider the finite measureable partition $\{D_n\}_{n\in\mathcal{I}}$ of D w.r.t. $\mathbb{B} = \bigcup_k \Omega_k$ as given in lemma 6.1. Notice that the case may occur, in which some D_n, $n \in \mathcal{I}$, are of measure zero. Since Lebesgue integral "ignores" zero sets, we can clearly throw out those part in (6.6). Hence, when applying lemma 6.3, we can assume w.l.o.g. that the partition of D w.r.t. $\mathbb{B}$ contains only subsets of D of positive measure.*

Remark 6.5. *Fix an $x \in D$. It is not hard to see, that since the image of each $\mathbf{f} \in L^2(D)^{\oplus K}$ lies in $\mathbb{C}^K$, the inequality:*

$$A_x\|\boldsymbol{a}\|_{\mathbb{C}^M} \leqslant \langle \mathrm{H}(x)\mathrm{H}(x)^*\boldsymbol{a}, \boldsymbol{a}\rangle_{\mathbb{C}^K} \leqslant B_x\|\boldsymbol{a}\|_{\mathbb{C}^M}, \quad \boldsymbol{a} \in \mathbb{C}^M, \tag{6.7}$$

is equivalent with the following condition concerning to elements of $L^2(D)^{\oplus K}$, for fixed $x \in D$:

$$A_x\|\mathbf{f}(x)\|_{\mathbb{C}^M} \leqslant \langle \mathrm{H}(x)\mathrm{H}(x)^*\mathbf{f}(x), \mathbf{f}(x)\rangle_{\mathbb{C}^K} \leqslant B_x\|\mathbf{f}(x)\|_{\mathbb{C}^M}, \quad \mathbf{f} \in L^2(D)^{\oplus K}, \tag{6.8}$$

with A_x and B_x are (possibly infinite) non-negative constants. It is obvious that the constants A_x and B_x fulfilling (6.7) can be given explicitly as:

$$A_x = \lambda_{\min}(\mathrm{H}(x)\mathrm{H}(x)^*) = (\sigma_{\min}(\mathrm{H}(x)))^2, \quad \text{and } B_x = \lambda_{\max}(\mathrm{H}(x)\mathrm{H}(x)^*) = (\sigma_{\min}(\mathrm{H}(x)))^2. \tag{6.9}$$

By the equivalence between (6.8) and (6.7), and by taking the smallest possible value of A_x and the biggest possible value of B_x, over $x \in D$, we obtain the constants:

$$A = \min_{n\in\mathcal{I}} \operatorname*{ess\,inf}_{x\in D_n} \lambda_{\min}(\mathrm{H}(x)\mathrm{H}(x)^*), \quad B = \max_{n\in\mathcal{I}} \operatorname*{ess\,sup}_{x\in D_n} \lambda_{\max}(\mathrm{H}(x)\mathrm{H}(x)^*), \tag{6.10}$$

for which, $\forall n \in \mathcal{I}$: $A\,Id \leqslant \mathrm{H}(x)\mathrm{H}(x)^* \leqslant B\,Id$, on a.e. $x \in D_n$. Furthermore, it is not hard[12] to see that by the choices of A and B as given in (6.10), the inequalities given in (6.4) is tight, in the sense that there exists signals f, for which the lower (resp.) bound is achieved.

So, it is now obvious that the right (resp. left) inequality in (6.5) is fulfilled if $B < \infty$ (resp. $A > 0$), or equivalently, the matrix $\mathrm{H}(x)$ contains finite entries (resp. $\mathrm{H}(x)$ has full-rank), for a.e. $x \in D_n$, $n \in \mathcal{I}$, and consequently for a.e. $x \in D$.

Let the constants A (resp. B), given in (6.10) is non-zero (resp. finite). The upper - and lower bound of the norm of the operator V^* is related to the constants A and B. Indeed, setting the inequality (6.5) in (6.6), one obtains:

$$A\|\mathbf{f}\|^2 \leqslant \|\mathrm{V}^*\mathbf{f}\|^2 \leqslant B\|\mathbf{f}\|^2, \quad \forall \mathbf{f} \in L^2(D)^{\oplus K}. \tag{6.11}$$

Hence the operator norm of V^* is lower (resp. upper) bounded by $\sqrt{A}$ (resp. $\sqrt{B}$), or equivalently by the essential minimum (resp. maximum) of the smallest (resp. biggest) singular value of $\mathrm{H}(x)$ over $x \in D$.

[12]Let $\tilde{D}$ be a subset of D of positive measure, s.t. $\lambda_{\min}(\mathrm{H}(x)\mathrm{H}(x)^*) = A$. For $f = \chi_D \in L^2(D)$, the lower bound is achieved. By the similar way, one can show that the upper bound is tight

Remark 6.6. *Now, let $M = K$. By noticing that the eigenvalues of* $\mathrm{H}(x)^*\mathrm{H}(x)$ *is simply the square of the modulus the eigenvalues of* $\mathrm{H}(x)$*, and by similar argumentation as in remark 6.5, we obtain the constants:*

$$\sqrt{A} = \min_{n\in\mathcal{I}} \operatorname*{ess\,inf}_{x\in D_n} |\lambda_{\min}(\mathrm{H}(x))|, \quad \sqrt{B} = \max_{n\in\mathcal{I}} \operatorname*{ess\,sup}_{x\in D_n} |\lambda_{\max}(\mathrm{H}(x))|, \tag{6.12}$$

for which $A\,Id \leqslant \mathrm{H}(x)^*\mathrm{H}(x) \leqslant B\,Id$, on a.e. $x \in D_n$, $n \in \mathcal{I}$, holds. Furthermore, the operator norm of V*, and accordingly, the operator norm of* V^* *is lower (resp. upper) bounded by A (resp. B).*

6.2. Multiple Fundamental Domains and induced isometric isomorphism

This section is devoted to established an isometric isomorphism between $L^2(\mathbb{B})$ and $L^2(D)^{\oplus K}$, where D is a "nice" fundamental domain of a uniform lattice Λ, and $\mathbb{B}$ a K-fundamental domain of Λ, which is by definition the union of the collection of mutually disjoint fundamental domains $\{\Omega_k\}_{k\in[K]}$. The corresponding isometric isomorphism is basically the direct sum of the Koopman operators $U_{\tau_{D\to\Omega_k}}$: $L^2(\Omega_k) \to L^2(D)$, $f \mapsto f \circ \tau_{D\to\Omega_k}$, $k \in [K]$. It is clear that since $\{\Omega_k\}_{k\in[K]}$ are mutually disjoint, $\{L^2(\Omega_k)\}_{k\in[K]}$, each seen as a closed subspace of $L^2(\mathbb{B})$, are mutually orthogonal. Hence $L^2(\mathbb{B})$ is isometric isomorphic to $\bigoplus_{k\in[K]} L^2(\Omega_k)$.

In the following, the corresponding lemma is given:

Lemma 6.7. *Let $\mathcal{G}$ be an LCA group. Given a countable uniform lattice Λ in $\mathcal{G}$. Furthermore, Let D be a fundamental domain of Λ, and $\mathbb{B}$ be a union of the mutually disjoint fundamental domains $\{\Omega_k\}_{k\in[K]}$ of Λ. Given the mapping:*

$$\mathcal{T}f := \begin{pmatrix} f \circ \tau_{D\to\Omega_1} \\ \vdots \\ f \circ \tau_{D\to\Omega_K} \end{pmatrix}, \quad \text{for } f \in L^2(\mathbb{B}). \tag{6.13}$$

Then $\mathcal{T}$ is a unitary equivalence between $L^2(\mathbb{B})$, and $L^2(D)^{\oplus K}$. Furthermore, the inverse of $\mathcal{T}$ is given explicitly by by:

$$L^2(D)^{\oplus K} \ni \tilde{f} \mapsto \sum_{k\in[K]} \tilde{f}_k \circ \tau_{\Omega_k\to D} \in L^2(\mathbb{B}). \tag{6.14}$$

Proof. By the discussion made in the beginning of this section, we already know that $L^2(\mathbb{B})$ is isometric isomorphic to $\oplus_{k\in[K]} L^2(\Omega_k)$ by the canonical isometric isomorphism (call Ψ):

$$L^2(\mathbb{B}) \ni f \mapsto (f\chi_{\Omega_1}, \ldots, f\chi_{\Omega_K}) \in L^2(D)^{\oplus K}.$$

Hence, to show that $\mathcal{T}$ is an isometric isomorphism, it is sufficient to show that $\oplus_{k\in[K]} L^2(\Omega_k)$ is isometric isomorphic to $L^2(D)^{\oplus K}$ by the direct sum of operators $\oplus_{k\in[K]} U_{\tau_{D\to\Omega_k}}$. Indeed, since each $U_{\tau_{D\to\Omega_k}}$ is unitary, one obtains immediately the statement. We already know, that for each $k \in [K]$, the adjoint/inverse of $U_{\tau_{D\to\Omega_k}}$ is exactly $U_{\tau_{\Omega_k\to D}}$. Accordingly, the adjoint/inverse of $\oplus_{k\in[K]} U_{\tau_{D\to\Omega_k}}$ is the direct sum of operators $\oplus_{k\in[K]} U_{\tau_{\Omega_k\to D}}$. Thence $(\mathcal{T})^* = ([\oplus_{k\in[K]} U_{\tau_{D\to\Omega_k}}] \circ \Psi)^* = (\Psi^* \circ [\oplus_{k\in[K]} U_{\tau_{\Omega_k\to D}}])$, as desired. ∎

The following lemma gives a "specification" of the isometric isomorphism $\mathcal{T}$ given in (6.14) under some constraints on D and $\mathbb{B}$:

Lemma 6.8. *Let $\mathbb{B} \subset \mathcal{G}$ be a relatively compact K-Fundamental domain of a countable uniform lattice Λ, and D be a relatively compact fundamental domain of Λ. Then there exist a finite almost partition $\{D_n\}_{n\in\mathcal{I}}$ of D, s.t. to each D_n, $n \in \mathcal{I}$, there correspond a finite sequence $\{\lambda_n^{(k)}\}_{k\in[K]}$ of elements of Λ*

s.t. the image of a function $f \in L^2(D)^{\oplus K}$ under the mapping $\mathcal{T}$, which is given in (6.14), can be written sectionwise as:

$$(\mathcal{T}f)(x) = \begin{pmatrix} f(x+\lambda_n^{(1)}) \\ \vdots \\ f(x+\lambda_n^{(K)}) \end{pmatrix}, \quad x \in D_n, \quad \forall n \in \mathcal{I}. \tag{6.15}$$

Furthermore, the inverse of $\mathcal{T}$ can in this case be given by:

$$L^2(D)^{\oplus K} \ni \tilde{f} \mapsto \sum_{n\in\mathcal{I}} \sum_{k\in[K]} (\tilde{f}_k \chi_{D_n})(\cdot - \lambda_n^{(k)}) \in L^2(\mathbb{B}). \tag{6.16}$$

Proof. Since $\mathbb{B}$ is a K-fundamental domain, it follows by definition, that $\mathbb{B}$ is the union of K mutually disjoint collection $\{\Omega_k\}_{k\in[K]}$ of fundamental domains of Λ. Since $\mathbb{B}$ is assumed to be relatively compact, it follows immediately that each Ω_k, $k \in [K]$, has to be relatively compact. Consider the finite measureable partition $\{D_n\}_{n\in\mathcal{I}}$ of D as suggested in lemma 6.1, and assign to each $n \in \mathcal{I}$, the corresponding sequence $\{\lambda_n^{(k)}\}_{n\in\mathcal{I}}$ of elements of Λ. Let $f \in L^2(D)^{\oplus K}$. It is now obvious, that for each $k \in [K]$, $f \circ \tau_{D\to\Omega_k}$ can be written sectionwise for each D_n, $n \in \mathcal{I}$ as $f(x+\lambda_n^{(k)})$, $x \in D_n$ (see the discussion about cross-transversals in the end of the subsection 4.2.1). Hence (6.15) holds. Furthermore, (6.16) is obvious to see. ∎

The following example helps to understand above concept:

Examples 6.9. *Consider the case given in examples 6.2. There we have:*

$$D_1 = [0, 0.34), \; D_2 = [0.34, 0.72), \; \text{and } D_3 = [0.72, 1).$$

To D_1, there corresponds lattice elements $\{\lambda_1^{(1)}, \lambda_1^{(2)}\} = \{2, 7\}$, to D_2, $\{\lambda_2^{(1)}, \lambda_2^{(2)}\} = \{1, 7\}$, and to D_3, $\{\lambda_3^{(1)}, \lambda_3^{(2)}\} = \{1, 6\}$. Hence for $f \in L^2(\mathbb{B})$, f can be written sectionwise as:

$$\begin{aligned} (\mathcal{T}f)(x) &= \begin{pmatrix} f(x+2) \\ f(x+7) \end{pmatrix}, \; x \in [0, 0.34), \; (\mathcal{T}f)(x) = \begin{pmatrix} f(x+1) \\ f(x+7) \end{pmatrix}, \; x \in [0.34, 0.72), \\ (\mathcal{T}f)(x) &= \begin{pmatrix} f(x+1) \\ f(x+6) \end{pmatrix}, \; x \in [0.72, 1). \end{aligned}$$

6.3. Main result

Let $\{\phi^{(m)}\}_{m\in[M]}$ be a collection of functions measureable on $\mathbb{B}$. We define formally the mapping $\tilde{\mathrm{V}} : L^2(D)^{\oplus M} \to L^2(\mathbb{B})$ by:

$$\mathbf{f} \mapsto \sum_{m\in[M]} \phi^{(m)} f_m^{\mathbb{B}}. \tag{6.17}$$

The mapping $\tilde{\mathrm{V}}$ gives information about the generating property of the sequence $\{\phi^{(m)} e_\lambda : m \in [M], \; \lambda \in \Lambda^\perp\}$ as obvious from the following computations of the image of the ONB in $L^2(D)^{\oplus K}$ as given in lemma 4.15:

$$\tilde{\mathrm{V}}\mathbf{e}_\lambda^{(m)} = \phi^{(m)} \chi_{\mathbb{B}} \sum_{\lambda\in\Lambda} \chi_D e_\lambda(\cdot - \lambda) = \phi^{(m)} \chi_{\mathbb{B}} e_\lambda, \quad m \in [M], \lambda \in \Lambda^\perp, \tag{6.18}$$

where the last equality follows from the fact that e_λ is Λ periodic. Hence, each elements of the collection $\{\phi^{(m)} e_\lambda \chi_{\mathbb{B}}\}_{m,\lambda}$ mentioned in the beginning of this chapter is simply the image of an element of the orthonormal basis for $L^2(D)^{\oplus K}$ under $\tilde{\mathrm{V}}$. If we are able to show that $\tilde{\mathrm{V}}$ is bounded and surjective (resp. bijective), then we can conclude that $\{\phi^{(m)} e_\lambda \chi_{\mathbb{B}}\}_{m,\lambda}$ is a frame (resp. Riesz basis) for $L^2(\mathbb{B})$. The following theorem gives the corresponding necessary and sufficient condition on the collection of functions $\{\phi^{(m)}\}_{m\in[M]}$:

Theorem 6.10. *Let Λ be an admissible uniform lattice in an LCA group $\mathcal{G}$, and D be a relatively compact fundamental domain of Λ, and $\mathbb{B}$ be a relatively compact K-fundamental domain of Λ. Given a collection of functions $\{\phi^{(m)}\}_{m\in[M]}$ on $\mathcal{G}$, whose restriction to $\mathbb{B}$ is measureable.*

Then, there exist a finite measureable partition $\{D_n\}_{n\in\mathcal{I}}$, where $\mathcal{I}$ is a finite index set, of D s.t. for each D_n there the corresponds a collection of elements $\{\lambda_n^{(k)}\}_{k\in[K]}$ of Λ s.t. the matrix Φ generated by $\{\phi^{(m)}\}_m$ given in (6.2) can be written sectionwise on each $\{D_n\}_{n\in\mathcal{I}}$ as:

$$\Phi(\omega) := \begin{pmatrix} \phi^{(1)}(x+\lambda_1^{(n)}) & \dots & \phi^{(M)}(x+\lambda_1^{(n)}) \\ \vdots & \ddots & \vdots \\ \phi^{(1)}(x+\lambda_K^{(n)}) & \dots & \phi^{(M)}(x+\lambda_K^{(n)}) \end{pmatrix}, \quad \text{a.e. } \omega \in D_n, \quad n \in \mathcal{I}. \tag{6.19}$$

Furthermore, the following equivalences hold true:

1. *$\{\phi^{(m)}e_\lambda\}_{m,\lambda}$ is a frame for $L^2(\mathbb{B})$ if and only if there exist constants $A, B > 0$ s.t. for all $n \in \mathcal{I}$, it holds:*
$$A\, Id \leqslant \Phi(x)\Phi^*(x) \leqslant B\, Id, \quad \text{a.e. } x \in D_n, \tag{6.20}$$

2. *Let $M = K$, $\{\phi^{(m)}e_\lambda\}_{m,\lambda}$ is a Riesz basis for $L^2(\mathbb{B})$, if and only if there exist constants $A, B > 0$ s.t. for all $n \in \mathcal{I}$:*
$$A\, Id \leqslant \Phi^*(x)\Phi(x) \leqslant B\, Id \quad \text{a.e. } x \in D_n. \tag{6.21}$$

Proof. Consider the isometric isomorphism $\mathcal{T}$ given in lemma 6.7. By lemma 6.8, there exist a measureable finite partition $\{D_n\}_{n\in\mathcal{I}}$ of D and to each D_n a lattice sequence $\{\lambda_n^{(k)}\}_{k\in[K]}$ s.t. $\mathcal{T}$ can be written sectionwise as:

$$(\mathcal{T}f)(\gamma) = (f(x+\lambda_n^{(1)}), \dots, f(x+\lambda_n^{(K)})), \quad x \in D_n, \quad n \in \mathcal{I}. \tag{6.22}$$

It is not hard to see[13] that for any $h \in L^2(D)$, and any $\lambda \in \Lambda$: $h^{\mathbb{B}}(\cdot - \lambda) = h$. So, we can compute the image of any $f \in L^2(D)^{\oplus M}$ under $\mathcal{T} \circ \tilde{\mathrm{V}}$, as follows:

$$\begin{aligned} ((\mathcal{T}\circ\tilde{\mathrm{V}})f)(x) &= \begin{pmatrix} \mathrm{V}f(x+\lambda_n^{(1)}) \\ \vdots \\ \mathrm{V}f(x+\lambda_n^{(K)}) \end{pmatrix} = \begin{pmatrix} \sum\limits_{k\in[M]} \phi^{(k)}(x+\lambda_n^{(1)})f_k(\gamma) \\ \vdots \\ \sum\limits_{k\in[M]} \phi^{(k)}(x+\lambda_n^{(K)})f_k(\gamma) \end{pmatrix} \\ &= \begin{pmatrix} \phi^{(1)}(x+\lambda_1^{(n)}) & \dots & \phi^{(M)}(x+\lambda_1^{(n)}) \\ \vdots & \ddots & \vdots \\ \phi^{(1)}(x+\lambda_K^{(n)}) & \dots & \phi^{(M)}(x+\lambda_K^{(n)}) \end{pmatrix} \begin{pmatrix} f_1(x) \\ \vdots \\ f_M(x) \end{pmatrix}, \end{aligned}$$

where $x \in D_n$, and $n \in \mathcal{I}$. So the operator $\mathrm{V} := \mathcal{T} \circ \tilde{\mathrm{V}}$ is simply a matrix multiplication by Φ, which is defined sectionwise as written in (6.19). From lemma 6.3, it follows that V is bounded and surjective (resp. bijective in case $K = M$) if and only if $\Phi(x)\Phi(x)^* \asymp \mathrm{Id}$ (resp. $\Phi(x) \asymp \mathrm{Id}$, in case $K = M$), for a.e. $x \in D_n$ and for all $n \in \mathcal{I}$. From lemma 2.3, we know that V is bounded and surjective (resp. bijective) if and only if $\tilde{\mathrm{V}}$ is bounded and surjective (resp. bijective). Finally, from lemma 2.12, it follows that $\tilde{\mathrm{V}}$ is bounded and surjective (resp. bijective) if and only if $\{\tilde{\mathrm{V}}\mathbf{e}_\lambda^{(m)}\}_{m,\lambda}$ is a frame (resp. Riesz basis) for $L^2(\mathbb{B})$, where $\{\mathbf{e}_\lambda^{(m)}\}_{\lambda,m}$ is the ONB of the direct sum of Hilbert space $L^2(D)^{\oplus K}$ as asserted from lemma 4.15.

We have already seen in (6.18) that the image of each $\{\mathbf{e}_\lambda^{(m)}\}_{m,\lambda}$ under $\tilde{\mathrm{V}}$ are exactly $\{\phi^{(m)}e_\lambda\}_{m,\lambda}$. Hence, the first (resp. second) equivalence is shown. ∎

[13]Follows from uniqueness of each lattice element and from the fact that D is a fundamental domain of Λ.

Notations 3. *For further approaches, the construction of the varying matrix given in (6.19) is worth noting: Given a fundamental domain D and a K-fundamental domain $\mathbb{B}$ = of an admissible uniform lattice Λ in $\mathcal{G}$. Further, given a collection of functions $\{\phi^{(m)}\}_{m\in[M]}$, each measureable on $\mathbb{B}$. Let $\{D_n\}_{n\in\mathcal{I}}$ be a finite measureable partition of D w.r.t. $\mathbb{B}$, as given in lemma 6.1. Further, assign to each D_n, $n \in \mathcal{I}$, the corresponding sequence $\{\lambda_n^{(k)}\}_{k\in[K]}$ from Λ.*

Define $\Phi : D \to \mathbb{C}^{K\times M}$ sectionwise as follows:

$$\Phi(x) := \begin{pmatrix} \phi^{(1)}(x+\lambda_1^{(n)}) & \dots & \phi^{(M)}(x+\lambda_1^{(n)}) \\ \vdots & \ddots & \vdots \\ \phi^{(1)}(x+\lambda_K^{(n)}) & \dots & \phi^{(M)}(x+\lambda_K^{(n)}) \end{pmatrix}, \quad x \in D_n, \quad n \in \mathcal{I}.$$

Then, we say Φ is the varying matrix on D generated by $\{\phi^{(m)}\}_m$. In case that clarity is necessary, we write Φ_ϕ^D. Notice that in this case, it is not necessary to mention $\mathbb{B}$, since it is implicitly[14] *given by $\{\phi^{(m)}\}_m$.*

Remark 6.11. *As follows from remark 6.5, the finite upper bound conditions (6.20) and (6.21) is surely ensured by choosing the collection of functions $\{\phi^{(m)}\}_{m\in[M]}$ s.t. they are each additionally essentially bounded on $\mathbb{B}$, i.e. $\phi^{(m)} \in L^\infty(\mathbb{B})$. Consider the varying matrix given in (6.19). Possible choices of the upper bound B and the lower bound A the condition (6.20) (resp. (6.21)) can now be given (see remark 6.5) as follows:*

$$A = \min_{n\in\mathcal{I}} \operatorname*{ess\,inf}_{\omega\in D_n} \lambda_{\min}(\Phi(x)\Phi(x)^*), \quad B = \max_{n\in\mathcal{I}} \operatorname*{ess\,sup}_{\omega\in D_n} \lambda_{\max}(\Phi(x)\Phi(x)^*), \tag{6.23}$$

and for condition (6.21), simply set $\Phi(x)^*\Phi(x)$ instead of $\Phi(x)\Phi(x)^*$ in (6.23). Of course, (6.23) can also be given alternatively by means of singular values of $\Phi(\cdot)$. Notice that the constants A and B given in (6.23) are tight bounds for the inequality (6.4), since, as we have seen in the proof of lemma 6.3, there exists a function f (resp. $\tilde{f}$)

Let $K = M$, as we have seen in remark 6.5, a possible choice of the bounds in (6.21) is:

$$\sqrt{A} = \inf_{n\in\mathcal{I}} \operatorname*{ess\,inf}_{x\in D_n} |\lambda_{\min}(\Phi(x))|, \quad \sqrt{B} = \sup_{n\in\mathcal{I}} \operatorname*{ess\,sup}_{x\in D_n} |\lambda_{\max}(\Phi(x))|.$$

Remark 6.12. *We have seen in remark 2.13 that the frame bounds of the frame generated by the operator $\tilde{\mathrm{V}}$ is related to the bounds of the operator norms of the adjoint operator $\tilde{\mathrm{V}}^*$. From remark 6.12, we already see that A (resp. B) given in (6.23) is the square of the lower (resp. upper) bound of $\|\tilde{\mathrm{V}}\|^*$. Hence from remark 2.13, A (resp B) is the lower (resp. upper) bound of the frame $\{\phi^{(m)}e_\lambda : m \in [M],\ \lambda \in \Lambda^\perp\}$.*

Suppose that we have a collection of functions $\{\phi^{(m)}\}_{m\in[M]}$ s.t. the condition (6.20) is fulfilled (call the exception set on each D_n, $n \in \mathcal{I}$, by E_n). By lemma 2.14, we can explicitly give a dual frame of the frame $\{\phi^{(m)}e_\lambda\}_{m,\lambda}$ for $L^2(\mathbb{B})$ generated by $\tilde{\mathrm{V}}$. Indeed, it follows from lemma 2.14, the canonical dual frame of $\{\phi^{(m)}e_\lambda\}_{m,\lambda}$ is generated by the Moore-Penrose pseudoinverse of $\tilde{\mathrm{V}}^*$, i.e. the operator $(\tilde{\mathrm{V}}\tilde{\mathrm{V}}^*)^{-1}\tilde{\mathrm{V}}$. Recall that $\tilde{\mathrm{V}} = \mathcal{T}^*\mathrm{V}$, where V is the operator corresponding to the varying matrix $\mathrm{G}(\cdot)$. So we obtain:

$$(\tilde{\mathrm{V}}\tilde{\mathrm{V}}^*)^{-1}\tilde{\mathrm{V}} = (\mathcal{T}^*\mathrm{V}\mathrm{V}^*\mathcal{T})^{-1}\mathcal{T}^*\mathrm{V}. \tag{6.24}$$

The mapping $\mathrm{V}\mathrm{V}^* : L^2(D)^{\oplus K} \to L^2(D)^{\oplus K}$ is given sectionwise by:

$$((\mathrm{V}\mathrm{V}^*)\mathbf{f})(x) = \Phi(x)\Phi(x)^*\mathbf{f}(x), \quad x \in D_n,\ n \in \mathcal{I}, \tag{6.25}$$

[14] $\mathbb{B}$ is the domain of each $\phi^{(m)}$

where $\mathbf{f} \in L^2(D)^{\oplus K}$. Since $\Phi(x)\Phi(x)^*$ is essentially invertible on each D_n, $n \in \mathcal{I}$, we can explicitly gives the inverse of VV^* essentially sectionwise as follows:

$$((\mathrm{VV}^*)^{-1}\mathbf{f})(x) = (\Phi(x)\Phi(x)^*)^{-1}\mathbf{f}(x), \quad x \in D_n \backslash E_n,\ n \in \mathcal{I}, \tag{6.26}$$

and the behaviour of $((\mathrm{VV}^*)^{-1}\mathbf{f})(x)$, on each E_n, $n \in \mathcal{I}$, can be arbitrarily chosen. Hence, we can continue the computation given in (6.24) as follows:

$$(\tilde{\mathrm{V}}\tilde{\mathrm{V}}^*)^{-1}\tilde{\mathrm{V}} = \mathcal{T}^*(\mathrm{VV}^*)^{-1}\mathrm{V}, \tag{6.27}$$

where $(\mathrm{VV}^*)^{-1}$ is given essentially as in (6.26).

By now, we know that the canonical dual frame of $\{\phi^{(m)}e_\lambda\}_{m,\lambda}$ consists of the image of the functions $\tilde{\boldsymbol{\phi}}_{m,\lambda}$, $m \in [M]$ and $\lambda \in \Lambda^\perp$, on $L^2(D)^{\oplus K}$ under the isometric isomorphism $\mathcal{T}^* : L^2(D)^{\oplus K} \to L^2(\mathbb{B})$, defined essentially as:

$$\tilde{\boldsymbol{\phi}}_{m,\lambda} := (\Phi(x)\Phi(x)^*)^{-1}\Phi(x)\mathbf{e}_\lambda^{(m)}(x), \quad x \in D_n \backslash E_n,\ n \in \mathcal{I}, \tag{6.28}$$

where λ (resp. n) goes over $\Lambda^\perp$ (resp. $[M]$). By defining the varying matrix:

$$(\Phi^*)^\dagger(x) := (\Phi(x)\Phi(x)^*)^{-1}\Phi(x), \quad x \in D_n \backslash E_n, \quad n \in \mathcal{I}, \tag{6.29}$$

where the subscript $(\cdot)^\dagger$ reminds us to the Penrose-Moore pseudoinverse of the matrix $(\cdot)$, we in the following shall see an alternative reformulation of the dual frame. For each $m \in [M]$ Since $\mathbf{e}_\lambda^{(m)}$ is zero on each entries, except on the m^{th}-entry, it yields: $[\tilde{\boldsymbol{\phi}}_{m,\lambda}]_k = [(\Phi^*)^\dagger]_{k,m}e_\lambda$. Hence, it yields by some elementary computations:

$$\mathcal{T}^*\tilde{\boldsymbol{\phi}}_{m,\lambda} = e_\lambda \mathcal{T}^*\boldsymbol{\varphi}^{(m)}, \tag{6.30}$$

where $[\boldsymbol{\varphi}^{(m)}]_k := [(\Phi^*)^\dagger]_{k,m}$, $k \in [K]$. Accordingly, we can denote the dual frame by $\tilde{\phi}_\lambda^{(m)}$ instead of $\phi_{m,\lambda}$

In the following thm., we summarize the discussions made above. Furthermore, we give the corresponding reconstruction formula for functions in $L^2(\mathbb{B})$.

Theorem 6.13. *Let D be a relatively compact fundamental domain of an admissible uniform lattice Λ in an LCA group $\mathcal{G}$, and $\mathbb{B}$ be a relatively compact K-fundamental domain of Λ. Further, assume that $\{\phi_\lambda^{(m)} : m \in [M],\ \lambda \in \Lambda\}$ is a frame for $L^2(\mathbb{B})$, where $\phi_\lambda^{(m)} := \phi^{(m)}e_\lambda$, $\forall m, \lambda$. For $m \in [M]$, let $\tilde{\phi}^{(m)}$ be defined as: $\tilde{\phi}^{(m)}(x) := (\mathcal{T}^*\boldsymbol{\varphi}^{(m)})$, where k^{th} entries, $k \in [K]$ of $\boldsymbol{\varphi}^{(m)}$ is given sectionwise by:*

$$[\boldsymbol{\varphi}^{(m)}(x)]_k = [(\Phi^*(x))^\dagger]_{k,m}, \quad x \in D_n \backslash E_n, \quad n \in \mathcal{I}, \tag{6.31}$$

with Φ is the varying matrix generated by $\{\phi^{(m)}\}_m$, as given in (6.19), and where for each $n \in \mathcal{I}$, E_n denotes the null subset of D_n, on which $\Phi(\cdot)\Phi(\cdot)^*$ is not invertible. Then the canonical dual frame of $\{\phi^{(m)}e_\lambda\}$ is exactly the collection consisting of $\tilde{\phi}_\lambda^{(m)} = e_\lambda\mathcal{T}^*\boldsymbol{\varphi}^{(m)}$, $\lambda \in \Lambda^\perp$, $m \in [M]$ where $\mathcal{T}^*$ is the adjoint of $\mathcal{T}$ as given in (6.16).

Furthermore, each $f \in L^2(\mathbb{B})$ can be expanded by the series:

$$f = \sum_{\substack{m\in[M]\\ \lambda\in\Lambda}} \langle f, \phi_\lambda^{(m)}\rangle \tilde{\phi}_\lambda^{(m)} = \sum_{\substack{m\in[M]\\ \lambda\in\Lambda}} \langle f, \tilde{\phi}_\lambda^{(m)}\rangle \phi_\lambda^{(m)}, \tag{6.32}$$

where the convergence is in L^2-norm.

Remark 6.14. *Let $M = K$, and suppose that $\{\phi^{(m)}e_\lambda\}_{m,\lambda}$ is a Riesz basis for $L^2(\mathbb{B})$. Then, it follows that for each $n \in \mathcal{I}$, the quadratic varying matrices $\Phi(x)$ generated by $\{\phi^{(m)}\}_m$, and $\Phi(x)^*$ are invertible* *a.e. on* $x \in D_n$*. Consequently,* (6.31) *reduces to the function defined sectionwise for each* $n \in \mathcal{I}$ *by:*

$$[\boldsymbol{\varphi}^{(m)}(x)]_k = [(\Phi^*(x))^{-1}]_{k,m}, \quad \text{a.e.} x \in D_n, \quad n \in \mathcal{I}, \tag{6.33}$$

Furthermore, $\{\tilde{\phi}^{(m)} e_\lambda\}_{m,\lambda}$*, as given in thm.* 6.13 *with* $\boldsymbol{\varphi}^{(m)}$ *as given above, is the unique dual of the Riesz basis* $\{\phi^{(m)} e_\lambda\}_{m,\lambda}$*. Of course, the reconstruction formula* (6.32) *holds in this case.*

We close this section by the following remark:

Remark 6.15. *Let* Ω *be a* K*-tiling subset of* $\mathcal{G}$ *by* Λ*, whose definition is given in def.* 4.6*. One can easily show that* Ω *can be written as the union of mutually disjoint subsets* $\{\Omega_k\}_{k\in[K]}$ *and* R*, where* Ω_k *is a almost fundamental domain of* Λ*, for each* $k \in [K]$*, and* R *is a set of measure zero in* $\mathcal{G}$*. So,* Ω *differs to a* K*-fundamental domain* $\mathbb{B}$ *by a zero set, which asserts that there exists a unitary equivalence between* $L^2(\Omega)$ *and* $L^2(\mathbb{B})$ *(call* $\Psi : L^2(\Omega) \to L^2(\mathbb{B})$*). By thm.* 6.10*, we may established a frame* $\{\phi^{(m)} e_\lambda\}_{\lambda,m}$ *for* $L^2(\mathbb{B})$*. By lemma* 2.11*, a frame for* $L^2(\Omega)$ *can be given by* $\{\Psi^*(\phi^{(m)} e_\lambda)\}_{m,\lambda}$*.*

6.4. Sampling and reconstruction of multi-banded signal with preprocessing

Since we have established sufficient and necessary condition on a collection of functions $\{\phi^{(m)}\}_{m\in[M]}$ measureable on $\mathbb{B}$ s.t. $\{\phi^{(m)} e_\lambda : m \in [M],\ \lambda \in \Lambda^\perp\}$ forms a frame for $L^2(\mathbb{B})$, and since we have given the corresponding canonical dual frame of $\{\phi^{(m)} e_\lambda\}_{m,\lambda}$, it is not a far step to give a sampling and reconstruction scheme of multi-banded finite-energy signals. The following theorem gives the corresponding modification for our purpose

Theorem 6.16 (Sampling and Reconstruction with preprocessing). *Let* $\mathcal{G}$ *be an LCA group. Given an admissible uniform lattice* $\tilde{\Lambda}$ *in* $\mathcal{G}^\wedge$*. Let* $\tilde{D} \subseteq \mathcal{G}^\wedge$ *be a relatively compact fundamental domain of* Λ*, and* $\tilde{\mathbb{B}}$ *be a relatively compact* K*-fundamental domain of* $\tilde{\Lambda}$ *in* $\mathcal{G}^\wedge$*. Given a collection of functions, whose frequency response is given by the collection of functions* $\{g^{(m)}\}_{m\in[M]}$ *measureable on* $\tilde{\mathbb{B}}$*. Forms by this collection the varying matrix* $\Phi_g(\cdot)$ *on* D *as given in notations* 3*.*

Suppose that the condition given in (6.20) *concerning to* $\Phi_g(\cdot)$ *is fulfilled, or equivalently that* $\{g^{(m)} e_\lambda\}$ *forms a frame for* $L^2(\tilde{\mathbb{B}})$*. Then, each functions* $f \in \mathcal{PW}^2_\Omega$*, can be reconstructed by the samples* $\{c^{(m)}_\lambda\}_{m,\lambda}$ *of* f*, preprocessed by the linear filters* $\{\overline{\check{g}^{(m)}}\}_{m\in[M]}$ *taken at each points of* $\Lambda^\perp$*, i.e.* $c^{(m)}_\lambda := (f * \overline{\check{g}^{(m)}})(\lambda)$*,* $m \in [M]$*,* $\lambda \in \Lambda^\perp$*, by the formula:*

$$f = \sum_{\substack{m\in[M] \\ \lambda\in\Lambda^\perp}} c^{(m)}_\lambda \mathcal{F}^{-1}\{\chi_{\tilde{\mathbb{B}}} \tilde{\phi}^{(m)}\}(\cdot - \lambda), \tag{6.34}$$

where $\tilde{\phi}^{(m)} := \boldsymbol{\varphi}^{(m)}$*,* $\forall m \in [M]$*, and each entry of* $\boldsymbol{\varphi}^{(m)}$ *is essentially defined sectionwise on finite measureable partitions* $\{\tilde{D}_n\}_{n\in\mathcal{I}}$ *of* $\tilde{D}$ *w.r.t.* $\tilde{\mathbb{B}}$ *by:*

$$[\boldsymbol{\varphi}^{(m)}(\gamma)]_k = [(\Phi^*_g(\gamma))^\dagger]_{k,m}, \quad \text{a.e.} \gamma \in D_n, \quad n \in \mathcal{I}. \tag{6.35}$$

Proof. Let $\{g^{(m)} e_\lambda\}_{m,\lambda}$ be a frame for $L^2(\tilde{\mathbb{B}})$. To show that this hold, thm. 6.10 asserts, that it is necessary and sufficient to show that the condition in (6.20) concerning to $\Phi_g(\cdot)$ is fulfilled. Furthermore, $\{g^{(m)} e_{-\lambda}\}_{m,\lambda}$ is also a frame for $L^2(\tilde{\mathbb{B}})$. Now, for each $f \in \mathcal{PW}^2_{\tilde{\mathbb{B}}}$, by means of thm. 6.13, and by noticing that frame is an unconditional basis, one can express $\hat{f}$ by the frame expansion:

$$\hat{f} = \sum_{\substack{m\in[M] \\ \lambda\in\Lambda^\perp}} c^{(m)}_\lambda e_{-\lambda} \mathcal{T}^* \boldsymbol{\varphi}^{(m)}, \tag{6.36}$$

where $\boldsymbol{\varphi}^{(m)}$ is given in (6.35), and $c_\lambda^{(m)} := \langle \hat{f}, \hat{g}^{(m)} e_{-\lambda} \rangle$, $\forall\ m \in [M]$, and $\lambda \in \Lambda^\perp$. Furthermore, the convergence is in $L^2(\tilde{\mathbb{B}})$ sense, and accordingly in $L^2(\mathcal{G}^\wedge)$. By inverse Fourier transforming the expression, and by noticing that each $c_\lambda^{(m)}$, $m \in [M]$, $\lambda \in \Lambda^\perp$, corresponds to the pointwise value of $f * \overline{g^{(m)}}$ on each λ, one gets the desired statements. ∎

Remark 6.17. *As similar as the discussion given in remark 6.14, in case $M = K$, and $\{g^{(m)} e_\lambda\}_{m,\lambda}$ is a Riesz basis for $L^2(\mathbb{B})$, $\boldsymbol{\varphi}^{(m)}$, $m \in [K]$ reduces to (6.33). Furthermore, the reconstruction formula (6.34) holds also with the corresponding dual Riesz basis.*

Remark 6.18. *By the similar argumentations made in Thm. 4.19, the right hand side on (6.34) gives a continuous function on $\mathcal{G}$, which vanishes at infinity. Furthermore, if the signal f of interests is continuous, one gets an equality instead of a.e. equality.*

In the following, we shall give a different way to see the reconstruction formula given in 6.34. Suppose that $\{g^{(m)} e_\lambda\}_{m,\lambda}$, and accordingly $\{g^{(m)} e_{-\lambda}\}_{m,\lambda}$, forms a frame for $L^2(\tilde{\mathbb{B}})$. Given a sequence $c_\lambda^{(m)} := (f * \overline{\breve{g}^{(m)}})(\lambda)$, $m \in [M]$, $\lambda \in \Lambda^\perp$, which consists of the samples of a signal $f \in L^2(\mathbb{B})$ filtered by the linear filters with frequency response $\overline{g^{(m)}}$. Recall that each $\{c_\lambda^{(m)}\}_{m,\lambda}$ is exactly the correlation of $\hat{f}$ with the frame element elements $\phi_\lambda^{(m)} := \tilde{\mathrm{V}} e_{-\lambda}^{(m)} = g^{(m)} e_{-\lambda}$. Hence by computations:

$$c_\lambda^{(m)} = \langle \hat{f}, \phi_\lambda^{(m)} \rangle = \langle \hat{f}, \tilde{\mathrm{V}} e_\lambda^{(m)} \rangle = \langle \tilde{\mathrm{V}}^* \hat{f}, e_\lambda^{(m)} \rangle.$$

So, from the samples $\{c_\lambda^{(m)}\}_{m,\lambda}$, we can reconstruct $\tilde{\mathrm{V}}^* \hat{f}$ by the orthonormal expansion:

$$h := \tilde{\mathrm{V}}^* \hat{f} = \sum_{\substack{m \in [M] \\ \lambda \in \Lambda^\perp}} c_\lambda^{(m)} e_\lambda^{(m)}.$$

Each m^{th}-entry of h can be seen as an analogy to the discrete-time Fourier transform of the sequence $\{c_\lambda^{(m)}\}$, which is frequently used in applications of electrical engineering. Since by definition, $\tilde{\mathrm{V}}^* = \mathrm{V}^* \mathcal{T}$, it is obvious that basically to reconstruct $\hat{f}$, we have to find $\tilde{f}$ for which $h = \mathrm{V}^* \tilde{f}$ holds. Once $\tilde{f}$ is obtained, $\hat{f}$ can be reconstructed from $\tilde{f}$ by computing $\hat{f} = \mathcal{T}^* \tilde{f}$. It is not hard to see, that since $\{g^{(m)} e_\lambda\}_{m,\lambda}$ is assumed to form a frame for $L^2(\mathbb{B})$ and accordingly V^* is left-invertible with inverse $(\mathrm{V}\mathrm{V}^*)^{-1}\mathrm{V}$. Hence, as we have seen previously, the task to solve $h = \mathrm{V}\tilde{f}$, reduces to the task of computing:

$$\tilde{f}(\gamma) = (\Phi(\gamma)\Phi^*(\gamma))^{-1} \Phi(\gamma) h, \quad \text{a.e. } \gamma \in \tilde{D}_n, \quad \forall n \in \mathcal{I}. \tag{6.37}$$

Let $K = M$, and assume that $\{g^{(m)} e_\lambda\}$ forms a Riesz basis for $L^2(\mathbb{B})$. It follows immediately that $\Phi(\gamma)$, and accordingly $\Phi(\gamma)^*$ is essentially invertible for a.e. $\gamma \in \tilde{D}_n$, $n \in \mathcal{I}$. Hence, $\tilde{f}$ in (6.37) can be computed more simple by:

$$\tilde{f}(\gamma) = (\Phi^*(\gamma))^{-1} h, \quad \text{a.e. } \gamma \in \tilde{D}_n, \quad \forall n \in \mathcal{I}. \tag{6.38}$$

Though, it might be possibly hard to compute above expression, since one might have to compute $(\Phi(\gamma)\Phi^*(\gamma))^{-1}$ (resp. $(\Phi(\gamma)^*)^{-1}$) for uncountably many points in $\tilde{D}$. In the next chapter, we will see that by a particular choice of the collection $\{g^{(m)}\}_m$, the expression (6.37) (resp. (6.38)) can be considerably simplified.

6.5. Robustness of the Sampling Scheme under presence of noise

Nevertheless, for a particular choice of linear filters $\{\overline{\breve{g}^{(m)}}\}_m$, one can gives a characterization of robustness of the sampling and reconstruction system under the presence of noise. Suppose that $\phi_\lambda^{(m)} := g^{(m)} e_\lambda$, $m \in [M]$, $\lambda \in \Lambda_{\mathcal{G}}^\perp$ forms a frame for $L^2(\mathbb{B})$, and let $\infty > A > 0$ (resp. $\infty > B > 0$) be the corresponding lower (resp. upper) frame bound. Denote the analysis operator corresponds to $\{\phi_\lambda^{(m)}\}_{m,\lambda}$ by $\mathrm{C} : L^2(\mathbb{B}) \to l^2$, $f \mapsto \{\langle f, \phi_\lambda^{(m)}\rangle\}_{m,\lambda}$. In this context, as we have already seen, C can be seen as the operator, which assigns each $f \in L^2(\mathbb{B})$ uniquely a sequence in $L^2(\mathbb{B})$. Since A and B are frame bounds of $\{\phi_\lambda^{(m)}\}_{m,\lambda}$, it follows immediately that $\sqrt{A}$ and $\sqrt{B}$ are upper - and lower bound of the operator norm of C, respectively. Finiteness of B, and positiveness of A, seen as bounds of the operator norm of C implies that there exists an inverse $\mathrm{C}^{-1} : l^2 \to L^2(\mathbb{B})$, which is simply the synthesis operator corresponding to the dual frame of $\{\phi_\lambda^{(m)}\}_{m,\lambda}$. It is not hard to show that the upper - and lower bound of the operator norm of C^{-1} is related to the upper - and lower bound of the norm of C by:

$$\frac{1}{\sqrt{B}} \leqslant \|\mathrm{C}^{-1}\| \leqslant \frac{1}{\sqrt{A}}. \tag{6.39}$$

Now, given $f \in \mathcal{PW}^2_{\mathbb{B}}$. Let $\mathbf{c} := \{c_\lambda^{(m)}\}_{m,\lambda} := \mathrm{C}f$ be the corresponding samples, assume that this samples is corrupted additively by the noise $\boldsymbol{\nu} := \{\nu_\lambda^{(m)}\}_{m,\lambda}$, where $\boldsymbol{\nu}$ is a sequence in l^2. We denote the signal-to-noise ratio by $\mathrm{SNR} := \|\mathbf{c}\|^2/\|\boldsymbol{\nu}\|^2$. We define specifically the corrupted sequences $\tilde{\mathbf{c}} := \tilde{c}_\lambda^{(m)}$ and the erronous signal $\tilde{f}$ by:

$$\tilde{c}_\lambda^{(m)} := c_\lambda^{(m)} + \nu_\lambda^{(m)}, \quad m \in [M],\ \lambda \in \Lambda^\perp, \quad \text{and} \quad \tilde{f} := \mathrm{C}^{-1}\{\tilde{c}_\lambda^{(m)}\}_{m,\lambda}. \tag{6.40}$$

So, it is not hard to compute the corresponding "relative square error":

$$\frac{\|\tilde{f} - \hat{f}\|^2}{\|\hat{f}\|^2} = \frac{\|\mathrm{C}^{-1}\tilde{\mathbf{c}} - \mathrm{C}^{-1}\mathbf{c}\|^2}{\|\hat{f}\|^2} = \frac{\|\mathrm{C}^{-1}\boldsymbol{\nu}\|^2}{\|\mathrm{C}\mathbf{c}\|^2} \leqslant \frac{\|\mathrm{C}^{-1}\|^2\|\boldsymbol{\nu}\|^2}{\|\mathrm{C}\|^2\|\mathbf{c}\|^2} \leqslant \frac{B}{A}\frac{1}{\mathrm{SNR}}. \tag{6.41}$$

It is obvious that from basic property of inverse fourier transform, that above bound gives also an upper bound of the error $\|f - \mathcal{F}^{-1}\tilde{f}\|^2/\|f\|^2$. Furthermore, by more detailed observation, one can see that above bounds is tight, in the sense that there exists signals $f \in L^2(\mathbb{B})$, and noise sequence $\boldsymbol{\nu} \in l^2$, s.t. the equality holds.

7. Application: Multicoset Sampling on LCA Groups of Multiband Signal

In this chapter, we show some possible choices of the collection of functions $\{\phi^{(m)}\}_{m\in[M]}$ on a K-fundamental domain $\mathbb{B}$ of Λ, which simplify in some sense, the analysis on the varying matrix on D generated by $\{\phi^{(m)}\}_m$, as given in (6.19). For convenient and for better understanding, we concern in this chapter simply Euclidean spaces $\mathbb{R}^N$, instead of general LCA groups. The results introduced in this chapter might also be abstracted to general LCA groups by some simple modifications. For easeness, we use the following notations

$$e_x(\cdot) := e(\langle x, \cdot\rangle) := e^{2\pi i\langle x,(\cdot)\rangle}. \tag{7.1}$$

If not otherwise stated, we follow the notations given in notations 1. As we know, since $(\mathbb{R}^N)^\wedge$ is identifiable with $\mathbb{R}^N$ (by the usual identification), we see both LCA groups as equal. Accordingly, the notation $(\mathbb{R}^N)^\wedge$ stands rather for the frequency domain, than the set of characters of $\mathbb{R}^N$. Since $(\mathbb{R}^N)^\wedge = \mathbb{R}^N$, it does not make sense to differ between Λ and $\tilde{\Lambda}$ (resp. D and $\tilde{D}$, $\mathbb{B}$ and $\tilde{\mathbb{B}}$). As usual, we part the "reference" fundamental domain D w.r.t $\mathbb{B}$ into the finite collection of measureable subsets $\{D_n\}_{n\in\mathcal{I}}$ as suggested by lemma 6.1, and assign to each D_n, $n \in \mathcal{I}$, the corresponding sequence $\{\lambda_n^{(k)}\}_{k\in[K]}$ from Λ. Furthermore, we can assume w.l.o.g. (see remark 6.4) that $\{D_n\}_{n\in\mathcal{I}}$ contains only sets of positive measure in $\mathbb{R}^N$.

In this chapter, we mainly concern ourselves with the following two choices of $\{\phi^{(m)}\}_m$:

(a) $\phi^{(m)} = e_{a^{(m)}}$, where $a^{(m)} \in \mathbb{R}^N$, $\forall m \in [M]$.

(b) $\phi^{(m)} = \psi((\cdot) - \beta^{(m)})$, where ψ is a function in $L^2(\mathbb{R}^N)$, and $\beta_m \in \mathbb{R}^N$, $\forall m \in [M]$.

We shall show , that the first choice is related to the so-called multi-coset sampling, and the second choice is related roughly to the so-called gabor sampling. We first begin by considering the choice $\phi^{(m)} = e_{a^{(m)}}$, $m \in \mathbb{R}^N$.

7.1. Frames of Exponentials and Multi-Coset Sampling

Given a collection of vectors $\mathbf{a} := \{a^{(m)}\}_{m\in[M]}$ in $\mathbb{R}^N$. To avoid trivialities, we assume that the elements of that collection differs pairwise. It is not hard to see that by the choices $g^{(m)} = e_{a^{(m)}}$, $\forall m \in [M]$, the varying matrix Φ on D generated by $\{\phi^{(m)}\}_m$ (see. notations 3) can be written as $\Phi = \mathrm{EU}$, where E is defined sectionwise, i.e. on each D_n, $n \in \mathcal{I}$ by:

$$\mathrm{E}(x) := \begin{pmatrix} e(\langle a^{(1)}, \lambda_1^{(n)}\rangle) & \dots & e(\langle a^{(M)}, \lambda_1^{(n)}\rangle) \\ \vdots & \ddots & \vdots \\ e(\langle a^{(1)}, \lambda_K^{(n)}\rangle) & \dots & e(\langle a^{(M)}, \lambda_K^{(n)}\rangle) \end{pmatrix}, \tag{7.2}$$

and U is a unitary diagonal matrix, given on whole D by:

$$\mathrm{U}(x) := \begin{pmatrix} e(\langle a^{(1)}, x\rangle) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e(\langle a^{(M)}, x\rangle) \end{pmatrix}, \quad x \in D_n. \tag{7.3}$$

In other words, Φ can be dissamble into a varying matrix E, which is sectionwise constant, and another varying matrix U, which is easy to handle, since it is diagonal, unitary, and defined on whole D (and not just sectionwise). So, by computations, the varying matrix $\Phi\Phi^*$ simplified to:

$$\Phi(x)\Phi(x)^* = \mathrm{E}(x)\mathrm{U}(x)\mathrm{U}(x)^*\mathrm{E}(x)^* = \mathrm{E}(x)\mathrm{E}(x)^*, \quad x \in D_n, \quad \forall n \in \mathcal{I}.$$

Since E is constant on each $n \in \mathcal{I}$, it stands clear to denote $\mathrm{E}^{(n)} := \mathrm{E}(x)$, $x \in D_n$, for all $n \in \mathcal{I}$. So, to check whether Φ fulfills the condition (6.20), it is enough to compute the singular values of all matrices in the finite collection $\{\mathrm{E}^{(n)}\}_{n\in\mathcal{I}}$, which is a fairly easy task since $\mathcal{I}$ is finite.

The computation of the canonical dual frame simplifies also by our choice of $\{\phi^{(m)}\}$: Suppose that $\{\phi^{(m)}_\lambda\}_{m,\lambda} := \{e_{a^{(m)}}e_\lambda\}_{m,\lambda}$ forms a frame for $L^2(\mathbb{B})$. By some computations, one can see that the canonical dual frame of $\{\phi^{(m)}_\lambda\}_{m,\lambda}$ is exactly the collection functions $\tilde{\phi}^{(m)}_\lambda = e_\lambda \mathcal{T}^* \boldsymbol{\xi}^{(m)}$, $m \in [M]$, where:

$$[\boldsymbol{\xi}^{(m)}(x)]_k := [(\mathrm{E}^{(n),*})^\dagger]_{m,k} e_{a^m}(x), \quad x \in D_n,\ n \in \mathcal{I}. \tag{7.4}$$

Especially, in the case $M = K$, the entries of $\boldsymbol{\xi}^{(m)}$, $m \in [M]$, is given sectionwise by

$$[\boldsymbol{\xi}^{(m)}(x)]_k := [(\mathrm{E}^{(n)})^\dagger]_{m,k} e_{a^{(m)}}(x), \quad x \in D_n,\ n \in \mathcal{I}. \tag{7.5}$$

Analogously, for the case $M = K$. For some choices of $\mathbf{a}$, to see whether $\{e_{a^{(m)}}e_\lambda\}_{m,\lambda}$ forms a Riesz basis for $L^2(\mathbb{B})$, remark 6.11 asserts, that one basically has to compute the modulus of the possible eigenvalue of the varying matrix Φ. In particular one simply has to compute the (modulus of the) eigenvalues of each matrices in the finite collection $\{E^{(n)}\}_{n\in\mathcal{I}}$, since Φ can be written as $\Phi = EU$, where E is a matrix varying only sectionwise, and U is a unitary varying matrix, and from the fact that unitary matrices preserves the modulus of eigenvalues, one simply has to compute the eigenvalues of the matrix in the finite collection $\{E^{(n)}\}_{n\in\mathcal{I}}$. Furthermore, one has even the following remarkable result:

Theorem 7.1. *Let $\Lambda \subset \mathbb{R}^N$ be a lattice, and D be a relatively compact fundamental domain of Λ. Then, for a.e. $a^{(1)}, \dots, a^{(K)} \in D$ the set*

$$\{e_{a^{(m)}-\lambda} : \lambda \in \Lambda^\perp,\ m \in [K]\} \tag{7.6}$$

is a Riesz basis for $L^2(\Omega)$ for any relatively compact K-fundamental domain $\mathbb{B} \in \mathbb{R}^N$

Proof. The only thing, which has to be checked, is that no eigenvalue of each matrix in the collection $\{E^{(n)}\}_{n\in\mathcal{I}}$ is zero, or equivalently, that each matrix in those collection is invertible. Define the trigonometric polynomial for $x_1, \dots, x_K \in \mathbb{R}^N$, and $\lambda_n \in \mathbb{R}^N$, $\forall k \in [K]$:

$$p(\{x_k\}_{k\in[K]}, \{\lambda_k\}_{k\in[K]}) := \sum_{\sigma\in S_K} \mathrm{sgn}(\sigma) \prod_{k=1}^{K} e(\langle x_{\sigma(k)}, \lambda_{\sigma(k)}\rangle), \tag{7.7}$$

where S_K denotes the permutation group of $[K]$, σ the corresponding permutation, and sgn the signature of a permutation. Notice that $p(\{a^{(m)}\}_{m\in[K]}, \{\lambda^{(n)}_m\}_{m\in[K]})$ is the determinant of E on D_n, $n \in \mathcal{I}$. Now, define the set of trigonometric polynomials:

$$\mathcal{P}_K := \{d(\{x_m\}, \{\lambda_m\}) : \lambda_m \in \Lambda,\ m \in [K]\}. \tag{7.8}$$

Obviously, $\mathcal{P}_K$ is countable since Λ is countable. Notice that since $\{\lambda_m\}_{m\in[K]}$ goes through Λ, $\mathcal{P}_K$ contains characteristic polynomials associated to other matrices E' corresponding to any K-fundamental domain of Λ. So appropriate choices of the set $\{a^{(m)}\}_{m\in[K]}$ are exactly those, which fulfills:

$$d(a^{(1)}, \cdots, a^{(K)}) \neq 0, \quad \forall d \in \mathcal{P}_K. \tag{7.9}$$

Since the zeros of a trigonometric polynomial is of measure zero, $\mathcal{P}_K$ is a countable set, and countable union of measure zero sets yields again a measure zero set, it follows that a.e. $(a^{(1)}, \ldots, a^{(K)}) \in \mathbb{R}^{N\times K}$ (call the exception set of measure zero $R \subset \mathbb{R}^{N\times K}$) is a "good" choice. Fix a fundamental domain D of Λ. Then for any generic $\{a_k\} \subset \mathbb{R}^K$ fulfilling (7.9), there exist unique $\{d_k\} \subset \tilde{D}$ s.t. $\{e_{a^{(m)}-\lambda}\}_{k,\lambda} = \{e_{d^{(m)}-\lambda}\}_{m,\lambda}$. So one can restrict the search for appropriate generic $\{a_k\}_k$ from $\mathbb{R}^N$ to $\tilde{D}$ (the exception set is also of measure zero since it is the intersection of R and a measureable subset D^K). ∎

Above thm. was mentioned in [33]. A general form of above thm. is given in [1].

Remark 7.2. *To find an "admissible" sampling cosets $\{\Lambda + a^{(m)}\}_{m\in[K]}$ for functions $f \in \mathcal{PW}^2_{\mathbb{B}}$, where $\mathbb{B}$ is any relatively compact fundamental domain of Λ, thm. 7.1 asserts to choose the each elements of the collection $\{a^{(m)}\}_{m\in[K]}$ by uniform distribution on a "nice" fundamental domain D. However, even if a.e. vector $a_1, \ldots, a_K \in D$ yields an admissible sampling cosets, the frame bounds concerning to the Riesz basis $\{e_{\lambda+a^{(m)}}\}_{\lambda,m}$ may be very bad, as asserted in [58].*

Now suppose that $\{e_{a^{(m)}}e_\lambda\}_{m,\lambda}$ forms a Riesz Basis for $L^2(\mathbb{B})$. The corresponding dual Riesz basis can explicitly be given by: $\tilde{\phi}^{(m)}_\lambda := e_\lambda\{\mathcal{T}^*\boldsymbol{\xi}^{(m)}\}$, $\lambda \in \Lambda^\perp$, and $m \in [M]$, where the entries of $\boldsymbol{\xi}^{(m)}$, $m \in [M]$, is given sectionwise by $[\boldsymbol{\xi}^{(m)}(x)]_k := [(\mathrm{E}^{*,(n)})^{-1}]_{m,k} e_{a^{(m)}}(x), \quad x \in D_n,\ n \in \mathcal{I}$.

In the following, we shall show the connection between above class of frame, and the sampling problem. Let $\{e_{a^{(m)}-\lambda}\}_{m,\lambda}$ be a frame for $L^2(\mathbb{B})$. Given a signal $f \in \mathcal{PW}^2_\Omega$. By some simple computations, it yields that $\langle \hat{f}, e_{a^{(m)}-\lambda}\rangle_{L^2(\mathbb{B})} = f(\lambda - a^{(m)})$, for each $\lambda \in \Lambda^\perp$, and $m \in [M]$. Hence, the frame expansion of f, as given in 6.34, can explicitly be computed by:

$$f = \sum_{\substack{\lambda\in\Lambda^\perp\\ m\in[K]}} f(\lambda - a_m)\frac{1}{m(D)}(\mathcal{F}^{-1}_{\mathbb{R}^N}\chi_{\mathbb{B}}\mathcal{T}^*\{\boldsymbol{\xi}^{(m)}\})(\cdot - \lambda), \tag{7.10}$$

where $\boldsymbol{\xi}^{(m)}$, $m \in [M]$ is given as in (7.4) and in case $M = K$, is given as in (7.5), and $\mathcal{F}^{-1}_{\mathbb{R}^N}$ denotes the usual inverse Fourier transform, i.e. $\mathcal{F}^{-1}_{\mathbb{R}^N} f = \int_{\mathbb{R}^N} f(\omega)e(\langle\omega, (\cdot)\rangle)$. Above sampling scheme is called multi-coset sampling, since the samples of the signal are taken on M-cosets of the subgroup $\Lambda^\perp$, i.e. $\Lambda^\perp + a^{(m)}$, $m \in [M]$. The corresponding illustration of the sampling device is given in fig. 7.1

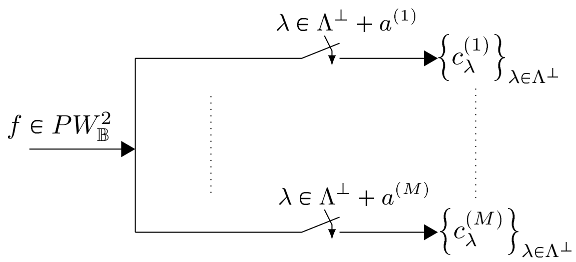


Figure 7.1.: Sampling device for multi-band multi-dimensional square-integrable signals induced by the frame $\{e_{a^{(m)}}e_\lambda\}_{\lambda,m}$.

For practical purposes (e.g.: [13], [61], [39], [40]), it is advantageous to consider the following type of $\mathbb{B}$:

Notations 4. *Let Λ be a lattice in $\mathbb{R}^N$, and D a relatively compact fundamental domain of Λ. we say $\mathbb{B} \subseteq \mathbb{R}^N$ is K-generated by the "elementar cell" D, if $\mathbb{B} = \bigcup_{k\in[K]} D + \lambda_k$, where $\{\lambda_k\}_{k\in[K]}$ is a finite collection of mutually distinguish elements of Λ.*

For the K-fundamental domain $\mathbb{B}$ of Λ, which is K-generated by a fundamental domain D, i.e. $\mathbb{B} = \bigcup_{k\in[K]} D + \lambda_k$, for some mutually distinguish λ_k, $k \in [K]$, the partition of D w.r.t. $\mathbb{B}$ contains only D, and the corresponding sequence of Λ is exactly $\{\lambda_k\}_{k\in[K]}$. Correspondingly, the isometric isomorphism between $L^2(\mathbb{B})$ and $L^2(D)^{\oplus K}$ given in lemma 6.8 simplified to $(\mathcal{T} f) = (f((\cdot) + \lambda_1), \dots, f((\cdot) + \lambda_K))$. The varying matrix E given in (7.2) is constant over D. Each of its entry has the form $[\mathrm{E}]_{k,m} = e(\langle a_m, \lambda_k\rangle)$. So for such a K-fundamental domain, it is fairly easy to check whether $\{e_{a_k - \lambda}\}_{k,\lambda}$ forms a frame for $L^2(\mathbb{B})$, for chosen $\{a_m\}_{m\in[M]}$, and to give the corresponding dual frame, which is necessary to establish the sampling and reconstruction formula given in (7.10). Furtermore, it is obvious to see that the alternative reconstruction method suggested at the end of section 6.4 reduces to a simple linear algebraic problem. Let $K = M$, for this class of K-fundamental domain, it is not hard to give an explicit choice of multi-coset sampling points:

Corollary 7.3. *Given a lattice Λ in $(\mathbb{R}^N)^\wedge$, and a fundamental domain D of Λ. Further, let $\mathbb{B}$ be a subset of $\mathbb{R}^N$, K-generated by D. Then there exists a particular choice of $a_1, \dots, a_K \in (\mathbb{R}^N)^\wedge = \mathbb{R}^N$ s.t. $\{\Lambda^\perp + a_m\}_{m\in[M]}$ are "admissible"[1] sample points for each functions in $\mathcal{PW}^2_{\mathbb{B}}$.*

Proof. Assume w.l.o.g. $K > 1$.We only need to choose $\{a^{(m)}\}_{m\in[M]}$ s.t. the matrix E, with entries $[\mathrm{E}]_{k,m} := e(\langle a_m\rangle)$, is invertible, or equivalently, its determinant is non-zero. Determine $a_0 \in \mathbb{R}^N$, s.t. $\nu_l \neq \nu_{l'}$, for each $l, l' \in [K]$, where $\nu_k := \langle a_0, \lambda_k\rangle$, $k \in [K]$. It is not hard to see that such choice is possible. Define $\nu_{\max} := \max\{|\nu_1|, \dots, |\nu_K|\}$. Let now $\{a_m\}_{m\in[M]}$ be given by:

$$a_m := \frac{(m-1)}{\nu_{\max} K} a_0, \quad \forall m \in [K].$$

Notice that, $|\langle a_2, \lambda_k\rangle| < 1/2K < 1/2$. By computations, it yields that the matrix E can be given as:

$$[\mathrm{E}]_{k,m} = e^{2\pi i \frac{\nu_k}{\nu_{\max}} \frac{m-1}{K}}, \quad k, m \in \{1, \dots, K\}.$$

Notice that E is a Vandermonde matrix, whose determinant is given by:

$$\det \mathrm{E} = \prod_{1 \leqslant l < l' \leqslant K} e\left(\frac{\nu_{l'}}{\nu_{\max}} \frac{1}{K}\right) - e\left(\frac{\nu_l}{\nu_{\max}} \frac{1}{K}\right) \tag{7.11}$$

Since we have ensured that $\nu_l \neq \nu_{l'}$, $\forall l, l' \in [K]$, $l \neq l'$, and $|\nu_k||\langle a_2, \lambda_k\rangle| < 1/2$, for each $k \in [K]$, and since $e(\cdot)$ is 1-periodic, and hence its value is determined uniquely in the interval $[-1/2, 1/2)$, above expression can not be zero. ∎

7.2. Gabor Systems

Another choice of the function $\{\phi^{(m)}\}_{m\in[M]}$ is given by some shifts of a function $\psi \in L^2(\mathbb{R}^N)$ to $\mathbb{B}$, i.e.:

$$\phi^{(m)} = \psi(\cdot - \beta^{(m)}), \quad m \in [M],$$

where $\beta^{(m)}$, $m \in [M]$ properly chosen constants, s.t. the varying matrix Φ on D generated by $\{\phi^{(m)}\}_{m\in[M]}$, fulfills the conditions given in thm. 6.10.

[1] Admissible means here, that $e_{a_m+\lambda}$, $m \in [K]$, $\lambda \in \Lambda^\perp$ forms a Riesz basis for $L^2(\mathbb{B})$.

The corresponding sampling scheme can be given as follows: Let $\{\phi^{(m)e_\lambda}\}_{m,\lambda}$ be a frame for $L^2(\mathbb{B})$, by a corresponding choice of $\psi \in L^2(\mathbb{B})$, and $\beta^{(m)} \in \mathbb{R}^N$, and given a function $f \in \mathcal{PW}^2_{\mathbb{B}}$. If we require additionally that $\beta^{(m)} \in \Lambda^\perp$, for each $m \in [M]$, we can compute the corresponding correlation between $\hat{f}$ and $\phi^{(m)}e_{-\lambda}$ for $m \in [M]$ and $\lambda \in \Lambda^\perp$ conveniently as follows:

$$\begin{aligned} c_\lambda^{(m)} &= \langle \hat{f}, \phi^{(m)} e_{-\lambda}\rangle_{L^2(\mathbb{B})} = \int\limits_{(\mathbb{R}^N)^\wedge} \hat{f}(\omega)\overline{\psi(\omega - \beta^{(m)})} e_\lambda(\omega) \mathrm{d}\mu_{(\mathbb{R}^N)^\wedge}(\omega) \\ &= \int\limits_{(\mathbb{R}^N)^\wedge} \hat{f}(\omega + \beta^{(m)})\overline{\psi(\omega)} e_\lambda(\omega + \beta^{(m)}) \mathrm{d}\mu_{(\mathbb{R}^N)^\wedge}(\omega) = \int\limits_{(\mathbb{R}^N)^\wedge} \hat{f}(\omega + \beta^{(m)})\overline{\psi(\omega)} e_\lambda(\omega) \mathrm{d}\mu_{(\mathbb{R}^N)^\wedge}(\omega) \\ &= ((f e_{-\beta^{(m)}}) * \overline{\check{\psi}})(\lambda), \end{aligned}$$

where the third equality follows from the translation invariance of Haar measure, and the fourth equality follows from the fact that e_λ, $\lambda \in \Lambda^\perp$ is Λ-periodic. The illustration of the proposed sampling scheme is illustrated in fig. 7.2. This sampling scheme may roughly be seen as the Gabor system for $\mathcal{PW}^2_{\mathbb{B}}$ with the window function ψ (see e.g. [18]).

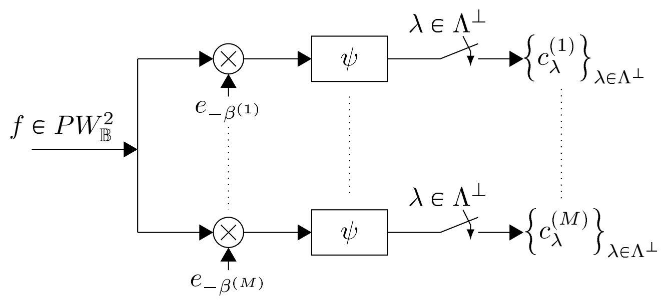


Figure 7.2.: Sampling device for multi-band multi-dimensional square-integrable signal induced by the frame $\{\psi(\cdot - \beta^{(m)})e_\lambda\}_{\lambda,m}$.

In the case that $\mathbb{B}$ is a K-generated by D, to choose such a function is fairly easy task: Let $M = K$, $\mathbb{B} = \cup_{k\in[K]} D + \lambda_k$. One can choose ψ, s.t. ψ is supported in D and ψ is non-zero a.e. on D, and the coefficients $\{\beta^{(m)}\}_{m\in[K]}$ can be chosen, s.t. $\beta^{(m)} := \lambda_m$. In this case, one immediately sees that the varying matrix Φ on D generated by $\{\phi^{(m)}\}_{m\in[M]}$ reduces to a diagonal matrix, whose diagonal is $[\Psi]_{m,m} = \phi^{(m)}$, $m \in [K]$. If ψ is in addition an ideal lowpass filter, the proposed sampling scheme is basically a classical bandpass sampling.

Moreover, it is unnecessary to require ψ to have its support in D. It is sufficient to require ψ to have its biggest value on some subset of D, and sufficiently rapid decay, s.t. the varying matrix Φ is diagonal dominant a.e. on D. In this case, Gershgorin's Thm. ensures that for a.e. $\omega \in D$, no eigenvalue of $\Phi(\omega)$ is zero, and accordingly Φ is a.e. invertible.

8. Summary, Discussions and Outlooks

The theory of locally compact Abelian groups provides a unifying approach to describe objects, which is of practical and theoretical relevance in communication engineering. Those included, in particular, the signals on a finite group of the form $\mathbb{Z}/k\mathbb{Z}$, which is rather known as array signals, for some $k \in \mathbb{N}$, the signals on the countable group $\mathbb{Z}$, which is rather known as discrete-time signals, and the signals on additive group $\mathbb{R}$, which is rather known as continuous-time signal. Furthermore, the theory of locally compact Abelian groups provides a "language", with which many other objects, one may face in electrical engineering, can also be handled. Those contain for example the signal on the product of the "elementary" LCA group.

By means of canonical sampling method for Paley-Wiener spaces, structured modulation, and finite-dimensional phase retrieval, we have shown that 2-D phase retrieval of infinite dimensional signals is possible, up to a certain exceptions. One may show that that exception is meagre in the considered Paley-Wiener space (as similar as it has been shown in [60, 48]). For a specific rectangular band-limit, we have shown that it is sufficient to have the rate 8-times the 2-D Nyquist rate. By choosing another choice of finite-dimensional phase retrieval e.g. [45], one may obtain a lower rate. It remains still an open problem, to determine the lowest possible rate one could achieve.

In this thesis, we have also concern ourselves with the so-called frames of exponentials weighted by a collection of functions. By some elementary approach, we have derived a necessary and sufficient conditions on the corresponding collection of functions, s.t. it forms a frame. In particular, it has been shown that the necessary and sufficient conditions is related to a varying matrix whose entry is determined by those collection of functions. We have shown, that this class of frame is related to the multi-channel sampling of finite-energy signals involving preprocessors on each of the channel. An appropriate formula has been given, with which the corresponding signal can be reconstructed from the samples obtained by the mentioned process. By the lack of the information on the structure, for general LCA group, the term, with which the sampling rate/density can be described, is not easy to handle, for detailed information, we refer to [19]. We have also shown an alternative way to reconstruct the signal, which leads to a linear algebraic problem. The latter has the potential to be applied to the problem of blind spectrum sensing, e.g. [13], [61], [39], [40].

Some examples in the Euclidean space $\mathbb{R}^N$ related to the previous frame construction has been given. In particular, those comprises the multi-coset sampling, and Gabor system. The corresponding sampling density (or following Landau's denotation: uniform density[1]) can be given: Since we have M sampling branches and each of the branches is sampled have the same sampling points $\Lambda^\perp$, the corresponding sampling density $\mathcal{D}_s$ of the considered system is simply M times the density $\mathcal{D}$ of the lattice $\Lambda^\perp$ (for the notion of lattice density, see [20]), which is in turn related to the Lebesgue measure of any[2] of relatively compact fundamental domain of its dual lattice Λ (e.g. take the measure of our "reference" fundamental domain D). Hence it yields: $\mathcal{D}_s = M\mathcal{D}(\Lambda) = Mm(D)$. In case that $M = K$, $\mathcal{D}_s$ coincides with the Lebesgue measure of the support $\mathbb{B}$ (which is the union of K mutually disjoint fundamental domains of Λ) of the signal of our interests, and accordingly optimal in Landau's sense.

For future works, one may apply above sampling scheme to the problem of blind spectrum sensing,

[1] We do not need to differ between upper- and lower uniform density, since in our case, it can be shown that both quantity coincides

[2] We have already shown that the measures of every relatively compact fundamental domain coincide

e.g. [13], [61], [39], [40] and also to the problem of phase retrieval. Furthermore, it is interesting to see how the proposed sampling scheme behaves for a bigger signal class e.g. see [5], [4]. Although, one may expect a negative answer, [41] may indicate that the multi-channel structure may be of advantage.

A. Topology

The following appendix, gives some basics on the field of topology. We shall give the results needed for our main approaches in a brief way. For detailed treatment, we refer to e.g. [6].

Given a set $\mathcal{X}$. Let $\mathcal{P}(\mathcal{X})$ denotes the power set of $\mathcal{X}$. In the following, we begin by introducing the notion of topology:

Definition A.1 (Topological space, relative topology). *Let $\mathcal{X}$ be a set, and let $\tau \subseteq \mathcal{P}(\mathcal{X})$ be a family of subsets of $\mathcal{X}$. τ is called a topology on $\mathcal{X}$, if $\emptyset \in \mathcal{X}$ and τ is closed under finite intersections and arbitrary union. Elements of τ are called open sets and their complement respectively closed sets. Furthermore, $(\mathcal{X}, \tau)$ is called topological space.*

Let $A \subset X$. The relative topology $\tau_{\mathcal{A}}$ of $\mathcal{A}$ induced by (X, τ) is defined as the intersection of elements of τ with $\mathcal{A}$.

If it is clear from the context, a topological space $(\mathcal{X}, \tau)$ is written simply as $\mathcal{X}$. Obviously, a subset of a topological space equipped with relative topology is itself a topological space. If not otherwise stated, $\mathcal{X}$ and $\mathcal{Y}$ are commited in the following to be a topological spaces.

Examples A.1. *The notion of metric spaces is assumed to be known. Let $(\mathcal{X}, d)$ be a metric space. One can canonically induce a topology as follows: Define the so-called open ball around $x \in \mathcal{X}$ with radius $\epsilon > 0$ by $\mathcal{B}_\epsilon(x) := \{y \in \mathcal{X} : d(x, y) < \epsilon\}$. The metrik d defines the topology τ_d by $O \in \tau_d \Leftrightarrow \forall x \in O \exists \epsilon > 0 : B_\epsilon(x) \subseteq O$, in other words, a subset is open if for each element of this subset there exist an open ball with sufficiently small radius around this element s.t. this open ball is contained in this subset. We call the topology on $\mathbb{R}^N$ induced by the euclidean metric as natural topology of $\mathbb{R}^N$*

In some cases, it is advantageous to describe a topology by means of smaller family of subsets:

Definition A.2 (Base of a topology). *Let $(\mathcal{X}, \tau)$ be a topological space. $\mathcal{B} \subseteq \tau$ is called base (or basis) for topology τ, if every $O \in \tau$ can be represented by union of elements of $\mathcal{B}$.*

A base $\mathcal{B}$ of a topology τ of a topological space $\mathcal{X}$ gives a fully description of τ, in the sense that a subset O of $\mathcal{X}$ is contained in τ if and only if O can be represented as disjoint union of elements of $\mathcal{B}$, or equivalently: $\forall x \in O, \exists B \in \mathcal{B}$, s.t. $x \in B \subseteq O$. So given just a base of a topology, one can decide, which subset of the considered set is contained in the topology (this justifies the idea behind the term "base"). We shall also sometimes say τ is *the topology induced by the base* $\mathcal{B}$. In particular, the natural topology of $\mathbb{R}^N$ has the following countable base: $\mathcal{B} := \{B_{1/m}(x) : x \in \mathbb{Q}, m \in \mathbb{N}\}$. Given a collection of subsets $\mathcal{B}$ of a set $\mathcal{X}$. The following theorem gives conditions on $\mathcal{B}$ s.t. one can surely reconstruct a topology on $\mathcal{X}$, for which $\mathcal{B}$ is a base:

Theorem A.2. *Let $\mathcal{X}$ be a set, and $\mathcal{B} \subseteq \mathcal{P}(\mathcal{X})$ a collection of subsets. Assume that $\mathcal{B}$ fulfills the following conditions:*

(a) $\mathcal{B}$ covers $\mathcal{X}$, i.e. $X = \bigcup_{B \in \mathcal{B}} B$,

(b) if $B_1, B_2 \in \mathcal{X}$, and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$, s.t. $x \in B_3 \subseteq B_1 \cap B_2$.

We define the interior -, boundary -, and closure of a subset in a topological space as follows:

Definition A.3 (Interior -, Boundary -, and Closure of a subset). *Let $\mathcal{X}$ be a topological space, and $A \subseteq \mathcal{X}$. The largest open set contained in A is called the interior of A and denoted by A°. The smallest closed set containing A is called the closure of A and denoted by $\overline{A}$. The boundary of A, denoted by ∂A, is defined as $\partial A := \overline{A} \cap \overline{A^c}$.*

"Nearness" is a qualitative notion in topology, which may be described by means of the following term:

Definition A.4 (Neighborhoods). *Let $N \subseteq \mathcal{X}$ be open. N is called neighborhood of $x \in \mathcal{X}$ if x lies in the interior of N, specifically:*

$$\exists O \subseteq \mathcal{X} \text{ open}: \quad x \in O \subseteq N.$$

The set of all neighborhood (or shortly: neighborhood system) of $x \in \mathcal{X}$ is denoted by $\mathcal{N}_x$.

So one can say in a sloppy way, that some elements of a topological space $\mathcal{X}$ is "near" to an element x, if they are contained in some neighborhoods of $\mathcal{X}$.

By means of the neighborhood system, one can formalize the term "isolatedness" of a point, and accordingly the term "discreteness" of a topological space:

Definition A.5 (Isolated Point and Discrete Topological Space). *Let $\mathcal{X}$ be a topological space. A point $x \in \mathcal{X}$ is said to be isolated, if there exists a neighborhood of x, which is only the singleton $\{x\}$. A topological space is said to be discrete, if each of its points is isolated.*

For an $x \in \mathcal{X}$, one can describe the neighborhood system of by smaller collection of neighborhoods of x, called neighborhood base:

Definition A.6 (Neighborhood Base). *Let $(\mathcal{X}, \tau)$ be a topological space, and $\mathcal{N}_x$ is the neighborhood system of $x \in \mathcal{X}$. A subcollection $\mathcal{B}_x \subseteq \mathcal{N}_x$ is called neighborhood base of x, if for every $N_x \in \mathcal{N}_x$, there corresponds a $B_x \in \mathcal{B}_x$ s.t.: $x \in B_x \subseteq N_x$.*

Again, a neighborhood base for a point $x \in \mathcal{X}$ costitutes a generating system for a neighborhood system of $x \in \mathcal{X}$, in the sense that one can decide by means of $\mathcal{B}$, whether a subset of the considered topological space is a neighborhood of x. Let τ be a topology on a set $\mathcal{X}$. We give in the following the connection between base for τ and neighborhood base for $x \in \mathcal{X}$:

- Let $\mathcal{B}$ be a base for τ, and U_x be a neighborhood of $x \in \mathcal{X}$, which is w.l.o.g. open (otherwise take an open O_x s.t. $x \in O_x \subseteq U_x$). Since U_x is open and $\mathcal{B}$ is a base, there clearly exists a subcollection of $\mathcal{B}$ whose union is U_x. Obviously, there exists an element of this subcollection, say B_x, with the property: $x \in B_x \subseteq U_x$, which asserts that B_x is an element of a (not yet specified) neighborhood base of x. For each U_x, form the collection $\mathcal{B}_x$ containing B_x. It is not hard to see that $\mathcal{B}_x$ is indeed a neighborhood base of x.

- Given a neighborhood base $\mathcal{B}_x$ for every $x \in \mathcal{X}$. Given a subcollection $\mathcal{B} \subseteq \tau$, fulfilling $\mathcal{B}_x \subseteq \mathcal{B}$, $\forall x \in \mathcal{X}$. For an open subset O of $\mathcal{X}$, notice that by the definition of neighborhood base, for each $x \in O$, there exists $B_x \in \mathcal{B}_x$ s.t. $x \in B_x \subseteq O$. So, one can write each open set O as $O = \bigcap_{x \in O} B_x$, which shows that every open sets is a union of elements of $\mathcal{B}$. Hence $\mathcal{B}$ is a base for τ.

By the same way as done to "reconstruct" the topology by means of its base, one can "reconstruct" the neighborhood system of a point by means of its neighborhood base. Given for each $x \in \mathcal{X}$ a collection of subsets $\mathcal{B}_x$. The following theorem gives conditions on each $\mathcal{B}_x$, $x \in \mathcal{X}$, s.t. one can give (or more conveniently: "reconstruct") a topology for which $\mathcal{B}_x$ is a neighborhood base of $x \in \mathcal{X}$:

Theorem A.3. *Let $\mathcal{X}$ be a set. Given for each $x \in \mathcal{X}$ a collection of subsets $\mathcal{B}_x$ for which the following conditions holds:*

- $\mathcal{B}_x \neq \varnothing$, and $B \in \mathcal{B}_x \Rightarrow x \in B \subseteq \mathcal{X}$,
- $B_1, B_2 \in \mathcal{B}_x \Rightarrow \exists B_3 \in \mathcal{B}_x$, *s.t.* $x \in B_3 \subseteq B_1 \cap B_2$
- $B \in \mathcal{B}_x \Rightarrow \exists A \subseteq \mathcal{X}$ *s.t.:*
  - $x \in A \subseteq B$,
  - $\forall y \in A : \exists B_y \in \mathcal{B}_y : y \in B_y \subseteq A$.

Define $\tau := \{O \subseteq \mathcal{X} : \forall x \in O,\ \exists B \in \mathcal{B}_x \text{ s.t. } x \in B \subseteq O\}$. *Then* τ *is a topology on* $\mathcal{X}$, *for which, for all* $x \in \mathcal{X}$, $\mathcal{B}_x$ *is a neighborhood base for* x.

As we will see later, if the considered space is a topological group, one can describe the topological property of the underlying space efficiently by means of the neighborhood system of the identity, and more efficiently by means of a neighborhood base of the identity.

The following classes of topological space is indispensable for analytic purposes:

Definition A.7 (Hausdorff -, First Countable -, and Second Countable Space). *Let* $\mathcal{X}$ *be a topological space.* $\mathcal{X}$ *is said to be a Hausdorff (topological) space, if* $\forall x, y \in \mathcal{X}$ *there exists a neighborhood* N_x *of* x *and a neighborhood* N_y *of* y *s.t.* $N_x \cap N_y = \varnothing$. $\mathcal{X}$ *is said to be a first (resp. second) countable space, if* $\mathcal{X}$ *(resp. any points of* $\mathcal{X}$*) possessess a countable (resp. neighborhood) base.*

Proposition A.4. *Let* $\mathcal{X}$ *be a discrete topological space. Then* $\mathcal{X}$ *is countable, if and only if* $\mathcal{X}$ *is second countable.*

Definition A.8 (Compact set, σ-Compact set, locally compact space). *Let* $A \subseteq \mathcal{X}$. A *is said to be compact if every family of open sets in* $\mathcal{X}$*, whose union contains* A *has a finite subfamily, whose union contains* A. $A \subseteq \mathcal{X}$ *is said to be* σ*-compact, if it can be written as a countable union of compact sets* $\mathcal{X}$ *is said to be locally compact, if every point of* $\mathcal{X}$ *has a compact neighborhood.*

Definition A.9 (Continuous map). *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a map.* f *is said to be continuous at* $x \in \mathcal{X}$ *if for every neighborhood* $N_{f(x)}$ *of* $f(x)$*, there exists a neighborhood* $N_x^{'}$ *of* x *s.t.* $f(N_x^{'}) \subseteq N_{f(x)}$. f *is said to be continuous on* $\mathcal{X}$ *if* f *is continuous at all* $x \in \mathcal{X}$.

Given a continuous mapping $f : \mathcal{X} \to \mathcal{X}^{'}$, and let $A \subseteq \mathcal{X}$. It is not hard to show that the restriction of f to A, i.e. the map $f|_A$, is continuous (of course, provided that A is equipped with the subspace topology). Several equivalent definitions of continuous map is summarized in the following proposition:

Proposition A.5. *Let* $(\mathcal{X}, \tau_{\mathcal{X}})$, $(\mathcal{Y}, \tau_{\mathcal{Y}})$ *be topological spaces, and* $f : \mathcal{X} \to \mathcal{Y}$ *a mapping. For* $x \in \mathcal{X}$*, let* $\mathcal{N}_x$ *be a neighborhood system of* x, $\mathcal{B}_x$ *be a neighborhood base of* x, $\mathcal{N}_{f_x}^{'}$ *be a neighborhood system of* $f(x)$*, and* $\mathcal{B}_{f(x)}^{'}$ *be a neighborhood base of* $f(x)$*. Then the following are equivalent:*

1. f *is continuous at* x
2. $\forall N \in \mathcal{N}_{f(x)}^{'} : f^{-1}(N) \in \mathcal{N}_x$
3. $\forall N \in \mathcal{N}_{f(x)}^{'} : \exists M \in \mathcal{N}_x : M \subseteq f^{-1}(N)$
4. $\forall B \in \mathcal{B}_{f(x)}^{'} : f^{-1}(B) \in \mathcal{N}_x$
5. $\forall B \in \mathcal{B}_{f(x)}^{'} : \exists M \in \mathcal{N}_x : f(M) \subseteq B$

In particular we shall often use, without mentioning, the equivalence (a)$\Leftrightarrow$(d) for our study on the topological structure of the dual group of an LCA group. Indeed it is easy to see that this equivalence holds true: Suppose that statement (d) holds true. By definition of the neighborhood base, any neighborhood N of $f(x)$ contains an element B of the neighborhood base of $f(x)$, and from (d) it follows that there exists a neighborhood M of x s.t. $f(M) \subseteq B \subseteq N$, which shows the desired statement.

Definition A.10 (Open - and Closed Map). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping between topological spaces. f is said to be an open (closed) map, if the image of all open (closed) sets in $\mathcal{X}$ is open (closed) in $\mathcal{Y}$, respectively.*

Definition A.11 (Homeomorphism). *A mapping $f : \mathcal{X} \to \mathcal{Y}$ is said to be a homeomorphism. If f is continuous, bijective, and its inverse is also continuous. X and Y are homeomorphic, if there exist a homeomorphism between them.*

Homeomorphism can alternatively defined as an bijective continuous open map, since openness and bijectivity of a mapping is equivalent to the continuity of its inverse. Let $\mathcal{X}$ and $\mathcal{Y}$ be in addition vector spaces. A mapping $f : \mathcal{X} \to \mathcal{Y}$ is said to be a linear homeomorphism if f is a homeorphism and linear map. It is clear, that f^{-1} is linear. Homeomorphism can also be equivalently described by means of the openness property:

Lemma A.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping between topological spaces. f is a homeomorphism if and only if f is bijective, continuous, and open.*

Definition A.12 (Connectivity). *$\mathcal{A} \subseteq \mathcal{X}$ is called connected if it can not be written as union of two disjoint sets, which are open in the relative topology.*

Definition A.13 (Discrete Set). *Let $\mathcal{X}$ be a topological set. A subset S is called discrete, if every $x \in S$ has a neighborhood N s.t. $S \cap N = \{x\}$. Every element of S is called isolated point.*

Definition A.14 (Product topology). *Let $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ be a family of topological spaces. The cartesian product of $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ can be equipped by the topology given by the base:*

$$\mathcal{B} := \{\prod_{i\in\mathcal{I}} O_i : O_i \subseteq \mathcal{X}_i \text{ open } \forall i \in \mathcal{I},\ \exists J \subseteq \mathcal{I} \text{ finite with } U_i = \mathcal{X}i\ \forall i \in \mathcal{I}\backslash J\}, \tag{A.1}$$

called the product topology.

Furthermore, we call the tuple consisting of cartesian product of topological spaces and the corresponding Given the cartesian product of topological spaces $\prod_{i\in\mathcal{I}} \mathcal{X}_i$, equipped with the product topology. We denote for $j \in \mathcal{I}$ the j-th projection map as $\mathrm{pr}_j : \prod_{i\in\mathcal{I}} \mathcal{X}_i \to \mathcal{X}_j$. One can show that by this choice of topology, pr_j is continuous for all $j \in \mathcal{I}$.

The most important result on product topologies is the following:

Theorem A.7 (Tychonoff Theorem). *Let $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ be a family of compact topological spaces, then the cartesian product of $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ equipped with the product topology is a compact topological space.*

By means of the product topology, one can induce a topology on the set of functions mapping between an arbitrary set and a topological space. Let $\mathrm{Map}(X, \mathcal{X})$ denotes the set of functions mapping from an arbitrary set X to a topological space $\mathcal{X}$. Clearly, one can see $\mathrm{Map}(X, \mathcal{X})$ as the cartesian power $\mathcal{X}^X$. Since $\mathcal{X}$ is a topological space, one can equip $\mathcal{X}^X$, and respectively $\mathrm{Map}(X, \mathcal{X})$ with the product topology, which is induced by the base containing sets of functions of the form:

$$\{f \in \mathcal{X}^X :\ f(x) \in U_x,\ \forall x \in F\}, \quad F \subseteq X \text{ finite, and } U_x \subseteq \mathcal{X} \text{ open},\ \forall x \in F. \tag{A.2}$$

Notice the equivalence of above expression with (A.1). This topology on function space is also called *pointwise convergence topology.* Let $\mathcal{X}$ be in addition compact. It follows immediately from Tychonoff theorem, that $\mathcal{X}^X$, and respectively $\mathrm{Map}(X, \mathcal{X})$, equipped with the product topology is compact.

Definition A.15 (Quotient topology). *Let $\mathcal{X}$ be a topological space, $\sim$ an equivalence relation on $\mathcal{X}$. Given $\mathcal{X}/\sim$ the set of equivalence classes in $\mathcal{X}$. Then the canonical mapping $p : \mathcal{X} \to \mathcal{X}/\sim$, $x \mapsto [x]$ is called the quotient mapping. The family of subsets:*

$$\tau^{\sim} := \{O \subseteq \mathcal{X}/\sim : p^{-1}(O) \text{ open in } \mathcal{X}\},$$

defines a topology in $\mathcal{X}/\sim$. The tuple $(\mathcal{X}/\sim, \tau^{\sim})$ is called quotient (topological) space.

The following theorem gives in some sense a clear description of continuous mapping:

Theorem A.8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, $\mathcal{X}/\sim$ be a quotient space with the canonical mapping p. For a mapping $\tilde{f} : \mathcal{X}/\sim \to Y$, it holds: $\tilde{f}$ is continuous (w.r.t. the quotient topology) if and only if $f := \tilde{f} \circ p : \mathcal{X} \to \mathcal{Y}$ is continuous.*

Above theorem shall not be hard to proof.

Definition A.16. *Let $\mathcal{X}$ be a topological space. Then:*

- *$\mathcal{X}$ is said to be first-countable, if there exists a countable base for the topology*
- *$\mathcal{X}$ is said to be second-countable, if there exists a countable neighborhood base for each point $x \in \mathcal{X}$*
- *$\mathcal{X}$ is said to be metrizable, if there exists a metric d, s.t. the topology is induced by d.*

Urysohn discovers the connection between second-countability and metrizability of a regular space: Every Hausdorff space, which is regular and second-countable, is metrizable.

B. MeasureTheory

In this appendix, we recall briefly, the basics of measure theory and Lebesgue integral. For a more detailed treatment, we refer to e.g. [14] and [10].

B.1. Basic Notions

We begin by introducing the system of subsets which constitutes the foundation of measure theory. Let $\mathcal{X}$ be a set and $\Sigma \subseteq \mathcal{P}(\mathcal{X})$ an subcollection of subsets of $\mathcal{X}$. Σ is said to be a *σ-algebra* of $\mathcal{X}$ if Σ fulfills the following properties:

1. $\varnothing \in \Sigma$.
2. $E \in \Sigma \Rightarrow E^c \in \Sigma$.
3. Σ is closed under countable union, i.e.: $E_1, E_2, ... \in \Sigma \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \Sigma$.

The tuple $(\mathcal{X}, \Sigma)$ containing a set $\mathcal{X} \neq \varnothing$ and a σ-algebra Σ of X is called measureable space. The following properties of σ-algebra are direct consequences of its definition:

- σ-algebra is closed under countable intersections
- σ-algebra is closed under set difference.

Given a measureable space $(\mathcal{X}, \Sigma)$, and a subset $E \subseteq X$. Then the collection of subsets of E, $\Sigma|_E := \{A \cap E : A \in \Sigma\}$, is called the σ-algebra of subset E of T. With $\Sigma|_E$, $(E, \Sigma|_E)$ becomes a measureable space in its own right. If not otherwise stated, we shall equip in this work every subset with the canonical measureable space structure inherited from their corresponding superset.

In many cases, it is impossible to write the σ-algebra of a corresponding space explicitly. However, it is sufficient to give the information about the structure of the σ-algebra by means of its so-called generator. From a given collection of subsets $\mathfrak{G}$ of $\mathcal{X}$, one can generate a σ-algebra as follows:

$$\sigma(\Sigma) = \cap\{\mathcal{A} \in \mathcal{P}(T) : \mathcal{A} \subseteq T,\ \mathcal{A}\ \sigma\text{-algebra}\}. \tag{B.1}$$

$\sigma(\mathfrak{G})$ denotes the σ-algebra generated by a collection of subsets $\mathfrak{G}$. $\mathfrak{G}$ is called the generator of $\sigma(\mathfrak{G})$. Notice that arbitrary intersection of σ-algebras of a set $\mathcal{X}$ gives a σ-algebra of $\mathcal{X}$, which implies that $\sigma(\mathfrak{G})$ is indeed a σ-algebra, and the generator of a σ-algebra is properly defined. Given a σ-algebra Σ of a measure space $\mathcal{X}$, generated by $\mathfrak{G} \subset \mathcal{P}(\mathcal{X})$, i.e. $\Sigma := \sigma(\mathfrak{G})$. It can be shown that the σ-algebra $\Sigma|_A$ of a subset $A \subseteq \mathcal{X}$ can be generated by $\mathfrak{G}|_A := \{E \cap \mathcal{A} : E \in \mathfrak{G}\}$.

In case that the considered space is topological, one can canonically generate the corresponding σ-algebra as follows: Let $(\mathcal{X}, \tau)$ be a topological space. A σ-algebra Σ of $\mathcal{X}$ is called Borel σ-algebra, if $\Sigma := \sigma(\tau)$. If $A \in \Sigma$ is open/closed w.r.t. the topology, then A is said to be Borel open/closed subset, respectively. Sets, which are measureable w.r.t. the Borel σ-Algebra, are exactly those, which can be represented as countable union of open sets, or as countable intersection of open sets. We will denote the Borel σ-algebra by $\mathfrak{B}$. When it is required, we emphasize the fact that $\mathfrak{B}$ is generated by topology τ by the notation $\mathfrak{B}(\tau)$. In case that $\mathcal{X}$ possesses some additional topological structures, Borel σ-algebra of $\mathcal{X}$ has also some alternative descriptions:

- If $\mathcal{X}$ is a metric space, Borel σ-algebra of $\mathcal{X}$ can be alternatively generated by the system of closed sets in $\mathcal{X}$.
- If $\mathcal{X}$ is a Hausdorff space, and there exists a countable sequence of compact sets $\{K_n\}$ s.t. $\mathcal{X} = \cup_n K_n$, i.e. $\mathcal{X}$ is compactly generated, then Borel σ-algebra of $\mathcal{X}$ can be generated by the system of compact subsets of $\mathcal{X}$.
- If $\mathcal{X}$ is a metric space possessing a countable basis, Borel σ-algebra of $\mathcal{X}$ can be generated by that basis.

If not otherwise stated, when considering topological spaces, we shall mostly equip those spaces with the corresponding Borel σ-algebra.

We are now ready to define measures properly as follows: Let (X, Σ) be a measureable set. A measure on X is defined as a function $\mu : \Sigma \to \overline{\mathbb{R}}$ fulfilling the following properties:

- $\mu \geqslant 0$
- $\mu(\varnothing) = 0$
- for pairwise disjoint countable families of sets $\{E_k\}$ contained in Σ, it holds: $\mu(\cup_k E_k) = \sum_k \mu(E_k)$.

The triple (X, Σ, μ) is called measure space. μ is called non-zero if $\mu(A) \neq 0$, $\forall A \in \Sigma$.

For each subset of a measureable space, one can canonically construct a measureable space: Let (X, Σ, μ) be a measure space, $A \subset X$ a subset. $(A, \Sigma|_A, \mu_A)$ is a measure space in its own right, where $\Sigma|_A$ denotes the σ-algebra on the subset A, and μ_A denotes the restriction of μ to A, which is defined as:

$$\mu_A(E) := \mu(A \cap E), \quad E \in \Sigma.$$

Furthermore, without mentioning, we equip the subset of a measure space with the canonical measure space structure inherited from its superset.

Especially, for Borel σ-algebra, the following classes of measures is desired to be given:

Definition B.1 (Regular Measure). *Let $(\mathcal{X}, \mathfrak{B}, \mu)$ be a Borel measure space. Then, μ is said to be regular, if μ fulfills the following conditions:*

1. *$K \subseteq \mathcal{X}$ compact $\Rightarrow \mu(K) < \infty$.*
2. *$A \in \mathfrak{B} \Rightarrow \mu(A) = \inf\{\mu(U) : A \subseteq U,\ U$ open$\}$.*
3. *$U \subseteq \mathcal{X}$ open $\Rightarrow \mu(U) = \sup\{\mu(K) : K \subseteq U,\ K$ compact$\}$.*

The second property in above definition is also called outer regularity, and the third inner regularity. Regular measure is, to say in a sloppy way, a measure for which the measure every measureable sets can be approximated from outer by open measurable sets and from inner by compact measureable sets. Furthermore, the following measure is canonical to a Borel measureable space:

Definition B.2 (Borel measure). *Let $(\mathcal{X}, \mathfrak{B})$ be a measure space, where $\mathfrak{B}$ is a Borel σ-algebra. A non-zero measure μ is said to be a Borel measure, if all the Borel subset is μ-measureable, and $\mu(A) < \infty$, for all compact $A \subseteq \mathcal{X}$.*

Some examples of measures which is of importance for us is given in the following:

- Let $A \in \mathcal{P}(\mathcal{X})$, where $\mathcal{X}$ is a countable set. μ which fullfills $\mu(A) = \#A$ if $\#A < \infty$ and $\mu(A) = \infty$ else, is a measure on X. Furthermore, μ is called counting measure.

- Lebesgue-Borel measure in $\mathbb{R}^N$, which assigns each element of Borel set in $\mathbb{R}^N$ the corresponding volume.

Let (X, Σ) and $(X^{'}, \Sigma^{'})$ be measureable spaces. A function $f : X \to Y$ is said to be measureable, if: $\forall E \in \Sigma^{'} : f^{-1}(E) \in \Sigma$. Notice that the measurability of a function depends only on the regarded σ-algebras. So it is unnecessary to define measures in order to check that property. Rather, it is sufficient to check the measurability of a function by observing the preimage of the sets generating the σ-algebra of the target space of that function, i.e. a function f mapping between measureable spaces (X, Σ) and $(X^{'}, \Sigma^{'})$, whose generator of its σ-algebra $\Sigma^{'}$ is $\mathcal{G}$, is measureable if and only if:

$$f^{-1}(G) \in \Sigma, \quad \forall G \in \mathcal{G}.$$

As an easy implication of above equivalent description of measureable functions, all continuous functions mapping between topological spaces, each is equipped with its corresponding Borel σ-algebra, are measureable. Furthermore it is not hard to see that (finite) composition of measureable functions yields again a measureable function.

B.2. Null Sets and Almost Everywhere Properties

As we shall soon see, the following type of set in a measure space is "negligible" in Lebesgue's sense:

Definition B.3 (Null-set, Full measure set). *Let $(\mathcal{X}, \Sigma, \mu)$ be a measure space. Then $A \in \Sigma$ is said to be a (μ-)null set, or set of measure zero w.r.t. μ, if $\mu(A) = 0$. A set $F \in \Sigma$ is said to be of full measure if its complement is a null set.*

We shall say simply null set, instead of μ-null set. In most cases, no confusion will be arised by this convention.

To avoid some troubles, it is convenient to extend a measure space in the following way:

Definition B.4 (Complete measure space). *A measure space $(\mathcal{X}, \Sigma, \mu)$ is said to be complete, if every subsets of μ-null set is contained in Σ and of measure zero. If $(\mathcal{X}, \Sigma, \mu)$ is complete, then μ is said to be complete.*

To make this concept more comprehensible, consider the measure space $\mathbb{R}^2$, equipped with the usual Borel σ-algebra, and the Lebesgue-Borel measure. Given a set A which is not measureable in $\mathbb{R}$, e.g. the Vitali set. The set $A \times \{0\}$ is not measureable w.r.t. to this measure space on $\mathbb{R}^2$, even though it is evident that this set has to be of measure zero, since $\mathbb{R}^2 \times \{0\}$ is of measure zero. So, it stands clear to "extend" the σ-algebra, s.t. it contains $A \times \{0\}$, and "assign" this set the measure zero. Given any measure space. It can be completed in the unique way by the following procedure:

Proposition B.1. *Let $(\mathcal{X}, \mathfrak{A}, \mu)$ be a measure space, and $\mathfrak{N}$ the system of all subsets of μ-null sets. Define:*

- $\tilde{\mathfrak{A}} := \{A \cup N : A \in \mathfrak{A},\ N \in \mathfrak{N}\}$
- $\tilde{\mu} : \tilde{\mathfrak{A}} \to \overline{\mathbb{R}}$, $\tilde{\mu}(A \cup N) := \mu(A)$, *for* $A \in \mathfrak{A}$, $N \in \mathfrak{N}$.

Then $\tilde{\mathfrak{A}}$ is a σ-algebra, $\tilde{\mu}$ well-defined, and $(\mathcal{X}, \tilde{\mathfrak{A}}, \tilde{\mu})$ is a complete measure space.

The Lebesgue measure in $\mathbb{R}^N$ can be seen as the unique extension of the Lebesgue-Borel measure in $\mathbb{R}^N$. The integral w.r.t. a measure μ turns out to be insensible to changes on the μ-null set, provided that the function is measureable. So, for purposes of measure theory, it is convenient to use the following term:

Definition B.5 (Almost everywhere property). *Let E be any reasonable property for elements of a measure space (X, Σ, μ). E is said to hold (μ-)almost everywhere (abbreviated μ-a.e.) if there exists a ($\mu-$) zero set $N \in \Sigma$ s.t. E holds for all $x \in N^c$.*

Some examples of the applications of almost everywhere property, which is used throughout this thesis, is given in the following:

- For functions $f, g : X \to Y$, It holds that $f = g$ a.e. on X if and only if there exist a zero set N s.t. $f|_{N^c} = g|_{N^c}$.
- A function $f : X \to \mathbb{F}$ is a.e. bounded if and only if there exists a positive scalar $\alpha \geqslant 0$ and set N of measure zero with $|f|_{N^c}| \leqslant \alpha$.

Proposition B.2. *Let (X, Σ, μ) be a complete measure space, and $f, g : X \to \mathbb{R}$ be functions, which agree pointwise almost everywhere on X. If f is measurable, then g is also measureable.*

B.3. Lebesgue Integral

Let $(\mathcal{X}, \Sigma, \mu)$ be a measure space. Consider a function $f : \mathcal{X} \to \mathbb{C}$. The idea behind the Lebesgue integral is to compute the integral of f by approximating it with the integral of in some sense "simple" functions. In the following, we shall illustrate the corresponding method more explicitly:

Call a function $\varphi : \mathcal{X} \to \mathbb{C}$ *simple*, if it is a finite linear combinations of characteristic functions of measureable subsets of $\mathcal{X}$, i.e. $\varphi = \sum_n \alpha_n \chi_n$, where $\{\alpha_n\}_n$ is a finite collection of elements of $\mathbb{C}$, and $\{E_n\}_n$ is a finite collection of measureable subsets of $\mathcal{X}$. The integral of such a simple function can easily be defined by: $\int_{\mathcal{X}} \phi(x)\mathrm{d}\mu = \sum_n \alpha_n \mu(E_n)$. The right side of the equality is defined, but may be equal to ∞. Furthermore, one can show that for another representation of ϕ, the integral of ϕ remains equal, which shows that the integral definition is appropriate. For a measureable, non-negative function $f : \mathcal{X} \to \overline{\mathbb{R}}$ one can show that there exists a monotone (increasing) sequence of simple functions $\{\phi_n\}_{n\in\mathbb{N}}$ converging pontwise to f. By the monotonicity of $\{\phi_n\}_{n\in\mathbb{N}}$, it follows that $\{\int_{\mathcal{X}} \phi_n \mathrm{d}\mu\}_{n\in\mathbb{N}}$ is also monotone, and converges in $[0, \infty]$. Thus, it stands clear to define the integral of such a non-negative function f by:

$$\int_{\mathcal{X}} f \mathrm{d}\mu := \lim_{n\to\infty} \int_{\mathcal{X}} \phi_n \mathrm{d}\mu. \tag{B.2}$$

Furthermore, for another monotone sequence of simple functions $\{\tilde{\phi}_n\}_{n\in\mathbb{N}}$ converging pointwise to f, the limits of the integral of the elements of that sequence coincide with above definition of the integral of f, which asserts that above definition makes sense. Now, we can say that a non-negative function is measureable, if its so-defined integral is finite. By analogue way, one can define integrability for general measureable function: A function $f : \mathcal{X} \to \hat{\mathbb{C}}$ is said to be (μ-)integrable (over $\mathcal{X}$) if f is measureable and if all of the four integrals:

$$\int_{\mathcal{X}} (\Re f)^{\pm} \mathrm{d}\mu, \quad \int_{\mathcal{X}} (\Im f)^{\pm} \mathrm{d}\mu,$$

are finite, where each of the above four integrals over non-negative functions is defined as in (B.2). So, the following expression:

$$\int_{\mathcal{X}} f \mathrm{d}\mu = \int_{\mathcal{X}} (\Re f)^{+} \mathrm{d}\mu - \int_{\mathcal{X}} (\Re f)^{-} \mathrm{d}\mu + i \int_{\mathcal{X}} (\Im f)^{+} \mathrm{d}\mu - i \int_{\mathcal{X}} (\Im f)^{-} \mathrm{d}\mu,$$

means the (μ-) integral of f over $\mathcal{X}$ or the Lebesgue integral of f over $\mathcal{X}$. The properties of Lebesgue integral can be looked up in standard textbooks, e.g. [14] and [10].

B.4. Main features of Lebesgue spaces

Let $(\mathcal{X}, \Sigma, \mu)$ be a measure space. For $0 < p \leqslant \infty$, we define the set $\mathcal{L}^p := \mathcal{L}^p(\mu) := \mathcal{L}^p(\mathcal{X}, \Sigma, \mu)$ as the set of measureable functions $f : \mathcal{X} \to \mathbb{C}$ for which the expression $\int_{\mathcal{X}} |f(x)|^p \mathrm{d}\mu(x)$ is finite, i.e for which $|f|^p$ is μ-integrable. It is convenient to associate the functional $\|\cdot\|_{\mathcal{L}^p} : \mathcal{L}^p \to \mathbb{R}_+$, $f \mapsto \int_{\mathcal{X}} |f(x)|^p \mathrm{d}\mu(x)$, with $\mathcal{L}^p$, $0 < p \leqslant \infty$. In case $p = \infty$, $\mathcal{L}^\infty$ denotes the set of all measureable functions $f : \mathcal{X} \to \mathbb{C}$ for which the associated functional $\|f\|_{L^\infty} := \operatorname{ess\,sup}_{x \in \mathcal{X}} |f(x)|$ is finite, i.e. $\mathcal{L}^\infty$ consists of functions, which are bounded almost everywhere. It is obvious that equipping $\mathcal{L}^p$ with the usual scalar multiplication and addition, $\mathcal{L}^p$, $\forall 0 < p \leqslant \infty$ becomes a vector space.

Some technical issues is needed to be concerned, when dealing with Lebesgue spaces: Notice that, for each $0 < p \leqslant \infty$, $\|\cdot\|_{\mathcal{L}^p}$ constitute just a semi-norm on $\mathcal{L}^p$ ($\|f\|_{\mathcal{L}^p} = 0$ for $f = 0$ a.e.), and furthermore, $\mathcal{L}^p$ equipped with $\|\cdot\|_{\mathcal{L}^p}$, cannot fulfill the Hausdorff separation axiom, since Lebesgue integral ignores null sets (functions which differs almost everywhere is not distinguishable by the topology induced from $\|\cdot\|_{\mathcal{L}^p}$). To overcome those unwished properties, one may consider the quotient $L^p = \mathcal{L}^p / N$ Instead of $\mathcal{L}_p$, where N denotes the set of measureable functions $f : \mathcal{X} \to \mathbb{C}$, which are $f = 0$ μ-almost everywhere. In other words, L^p consists of equivalence classes of functions, in which the elements differ on sets of measure zero. Equipping L^p with canonical quotient vector space structure induced from $\mathcal{L}^p$, and with the norm $\|[f]\|_{L^p} := \|f\|_{\mathcal{L}^p}$, one can show that L^p becomes a Banach space, for all $1 \leqslant p \leqslant \infty$, and in particular L^2 is a Hilbert space (Riesz-Fischer Thm. asserts that $\mathcal{L}^p$ is complete, for each $1 \leqslant p \leqslant \infty$, and one obtains immediately the same result for L^p, $1 \leqslant p \leqslant \infty$). Let Ω be a locally compact Hausdorff space, and let Ω be equipped with Radon measure. It is well-known that the space of continuous function with compact support $C_c(\Omega)$ is dense in $L^p(\Omega)$, for each $1 \leqslant p \leqslant \infty$.

Given a measure space $\mathcal{X}$, and a measureable subset $\Omega \subseteq \mathcal{X}$. Let $f : \Omega \to \mathbb{C}$ be given. A trivial extension of f to $\mathcal{X}$ is defined as the function $\tilde{f} : \mathcal{X} \to \mathbb{C}$, for which it holds: $\tilde{f}\Big|_{\Omega} = f$, and $\tilde{f}(x) = 0$, a.e. $x \in \mathcal{X} \backslash \Omega$. Let $0 < p \leqslant \infty$. Furthermore, $L^p(\Omega)$ can be seen as a subspace of the vector space $L^p(X)$, if each function $f \in L^p(\Omega)$ is trivially extended to $\mathcal{X}$. Furthermore, by this view, $L^p(\Omega)$ is closed in $L^p(\Omega)$. We shall use this issues throughout the thesis without mentioning. Now, consider again the Lebesgue space $L^2(\mathcal{X})$, and let $L^2(\Omega)$, and $L^2(\Omega')$ be closed subspace of $L^2(\mathcal{X})$. If Ω and Ω' are disjoint, then obviously, $L^2(\Omega)$ and $L^2(\Omega')$ are orthogonal subspaces of $L^2(\mathcal{X})$ (w.r.t. $\langle \cdot, \cdot \rangle_{L^2(\mathcal{X})}$). In addition, if the union Ω and Ω' coincides with $\mathcal{X}$, then $L^2(\mathcal{X})$ is isometric isomorph with the direct sum $L^2(\Omega)$

Let Ω and Ω' be measureable subsets of $\mathcal{X}$, which differs only in null set. There is an obvious isometric isomorphism between $L^2(\Omega)$ and $L^2(\Omega')$, viz. $f \mapsto f|_{\Omega'}$ if $\Omega' \subseteq \Omega$, and $f \mapsto \tilde{f}$, where $\tilde{f}$ is any extension of f, if $\Omega \subset \Omega'$.

B.5. Measure invariant mapping and induced Isometric Isomorphism

Definition B.6. *Let $(\mathcal{X}, \Sigma, \mu)$ and $(\mathcal{X}', \Sigma', \mu')$ be measureable spaces. A measureable mapping $\phi : \mathcal{X} \to \mathcal{X}'$ is said to be measure-preserving if:*

$$\forall E \in \Sigma' : \; \mu(\phi^{-1}(E)) = \mu'(E).$$

If ϕ is invertible, and its inverse ϕ^{-1} is a measure-preserving mapping, then ϕ is called invertible measure-preserving mapping.

It is not hard to see that the following holds true:

Proposition B.3. *Let $(\mathcal{X}, \Sigma, \mu)$ and $(\mathcal{X}', \Sigma', \mu')$ be measure spaces. Then the following are equivalent:*

(a) $\phi : \mathcal{X} \to \mathcal{X}'$ is a measure-preserving mapping.

(b) *For any* $f \in L^1(\mathcal{X}^{'})$, *it holds:*

$$\int_{\mathcal{X}'} f \mathrm{d}\mu^{'} = \int_{\mathcal{X}} f \circ \phi \mathrm{d}\mu$$

Proof. The implication "(b)⇒(a)", is easy to show by setting $f = \chi_A$, for any measureable subset A of $\mathcal{X}^{'}$, and noticing that $\chi_A \circ \phi = \chi_{\phi^{-1}(A)}$.

"(a)⇒(b)": It follows immediately from the definition of measure preserving mapping, that (b) holds for all characteristic function of the form χ_A, for $A \in \Sigma^{'}$. By some simple computation, the result holds also for simple functions. Now we show the implication for any $f \in L^1(\mathcal{X})$. Since f can be decomposed into real - and imaginary part, and in turn both parts can each be decomposed into positive and negative part, it is sufficient to show the result for integrable non-negative function $f \geqslant 0$, $f \in L^1(\mathcal{X}^{'})$. Let $\{u_n\}_{n\in\mathbb{N}}$ be an monotone sequence of simple functions on $\mathcal{X}^{'}$ which converges pointwise to f. Then it can be shown that $\{u_n \circ \phi\}_{n\in\mathbb{N}}$ is also an increasing sequence of simple function on $\mathcal{X}$ and that it converges pointwise to the non-negative integrable function $f \circ \phi \in L^1(\mathcal{X}^{'})$. Hence:

$$\int_{\mathcal{X}'} f \mathrm{d}\mu = \lim_{n\to\infty} \int_{\mathcal{X}'} f_n \mathrm{d}\mu = \lim_{n\to\infty} \int_{\mathcal{X}} f_n \circ \phi \mathrm{d}\mu = \int_{\mathcal{X}} f \circ \phi \mathrm{d}\mu, \quad \blacksquare$$

where the first - and the third equality follows from Monotone Convergence Theorem, and the second inequality follows from the fact that the implication "(a)⇒(b)" holds for any simple functions.

Proposition B.4. *Let* $(\mathcal{X}, \Sigma, \mu)$ *and* $(\mathcal{X}^{'}, \Sigma^{'}, \mu^{'})$ *be measure spaces. Given a measure-preserving mapping* $\phi : \mathcal{X} \to \mathcal{X}^{'}$. *For* $1 \leqslant p < \infty$, *define the mapping* U_ϕ *by:*

$$U_\phi(f) := f \circ \phi, \quad f \in L^p(\mathcal{X}^{'}).$$

Then U_ϕ *is an injective isometry from* $L^p(\mathcal{X}^{'})$ *onto a closed subspace of* $L^p(X)$.

Proof. Let $1 \leqslant p < \infty$. Given a function $f \in L^p(\mathcal{X}^{'})$. Then $|f|^p$ defines a function on $L^1(\mathcal{X}^{'})$, hence from prop. B.3, the following holds:

$$\int_{\mathcal{X}'} |f|^p \mathrm{d}\mu^{'} = \int_{\mathcal{X}} |f|^p \circ \phi \mathrm{d}\mu = \int_{\mathcal{X}} |f \circ \phi|^p \mathrm{d}\mu, \tag{B.3}$$

which shows that $U_\phi f = f \circ \phi$ is an element of $L^p(\mathcal{X})$. Since linearity of U_ϕ is obvious, we can conclude that U_ϕ is an operator between $L^p(\mathcal{X}^{'})$ and $L^p(\mathcal{X})$. Furthermore, (B.3) shows also that U_ϕ is an isometry, and hence a bounded operator of norm 1. In addition, the norm equivalence between $U_\phi f$, for all $f \in L^p(\mathcal{X})$ implies immediately that U_ϕ is injective (the inequality $\|f\| \leqslant c\|U_\phi f\|$ is fulfilled with $c = 1$) and has closed range (Bounded operators have always closed range). ■

To ensure that the Koopman operator related to a measure-preserving mapping is isometric isomorphic, one has the following sufficient condition:

Proposition B.5. *Let* $(\mathcal{X}, \Sigma, \mu)$ *and* $(\mathcal{X}^{'}, \Sigma^{'}, \mu^{'})$ *be measure spaces. Given a measure-preserving mapping* $\phi : \mathcal{X} \to \mathcal{X}^{'}$. *If* ϕ *is bijective, and its inverse* ϕ^{-1} *is measureable, then* U_ϕ *is a unitary equivalence between* $L^2(\mathcal{X}^{'})$ *and* $L^2(\mathcal{X})$. *Furthermore, the inverse is given by* $U_\phi^* = U_{\phi^{-1}}$.

Proof. One may simply modify Thm. 2.8 in [55] to our setting, and subsequently see that above statement is an implication of that. ■

C. Topological Groups

C.1. Basics on Group Theory

In the following, we recall some group theoretical basics indispensable for the main approach.

Definition C.1 ((Abelian) group). *Let $\mathcal{G}$ be a set, and $\circ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be an operation on $\mathcal{G}$. We say $(\mathcal{G}, \circ)$ is a group if $\circ$ fulfills the following properties:*

- *(Associativity) $\forall x, y, z \in \mathcal{G}$: $(x \circ y) \circ z = x \circ (y \circ z)$*
- *(Existence of a neutral element): $\exists e \in \mathcal{G}$: $e \circ a = a = a \circ e$*
- *(Existence of inverse): $\forall a \in \mathcal{G} : \exists a^{'} \in \mathcal{G}$: $a \circ a^{'} = e = a^{'} \circ a$.*

Furthermore, if $\circ$ is in addition commutative, i.e.:

$$\forall x, y \in \mathcal{G} : \quad x \circ y = y \circ x,$$

then $(\mathcal{G}, \circ)$ is called abelian.

The neutral element is also called identity or unit. If it is clear from the context, $(\mathcal{G}, \circ)$ is written simply $\mathcal{G}$. We say a group is a multiplicative/additive, if the corresponding operation is a multiplication/addition. We say $\mathcal{G}$ is a group w.r.t. the operation $\circ$ defined on $\mathcal{G}$, if $(\mathcal{G}, \circ)$ is a group. We say $\mathcal{G}$ is written multiplicatively/additively if the corresponding operation is written multiplicatively/additively, respectively. The unit in a group $\mathcal{G}$ written multiplicatively or additively is sometimes denoted by 1 or 0, respectively. If necessary, the unit of a group $\mathcal{G}$ is emphasized by the subscript $\mathcal{G}$. Unless otherwise stated, all the groups occuring in this section are written multiplicatively. For an element g of a group $\mathcal{G}$, the notation g^n, for $n \in \mathbb{N}$, stands commonly for the multiplication of n-copies of g, also called n-power of g, for $-n \in \mathbb{N}$, the multiplication of n-copies of g^{-1}, and especially for $n = 0$, g^n is equal to the identity of $\mathcal{G}$. Furthermore, the rule $g^m g^n = g^{m+n}$, for $m, n \in \mathbb{Z}$, holds. Analogously, the n-power of an element g of an additive group $\mathcal{G}$, denoted by ng, can be established.

For a finite sequences of groups, one can define the following group structure, suitable for the cartesian product structure:

Definition C.2 ((Outer) Direct Product). *Let $\{\mathcal{G}_n\}_{n \in [N]}$ be a finite sequence of groups, each written multiplicatively, where e_n is the identity of $\mathcal{G}_n$, for each $n \in [N]$. The cartesian product of those groups is defined by:*

$$\mathcal{G} := \prod_{n \in [N]} \mathcal{G}_n := \mathcal{G}_1 \times \cdots \times \mathcal{G}_N := \{(a_1, \ldots, a_N) : a_n \in \mathcal{G}_n, \ \forall n \in [N]\}.$$

$\mathcal{G}$ equipped with the composition written multiplicatively:

$$(a_1, \ldots, a_N)(b_1, \ldots, b_N) := (a_1 b_1, \ldots, a_N b_N),$$

the inversion:

$$(a_1, \ldots, a_N)^{-1} := (a_1^{-1}, \ldots, a_N^{-1}),$$

forms a group called (outer) direct product of $\{\mathcal{G}_n\}_{\in[N]}$, *where the corresponding identity is* $(e_1, \ldots, e_N)$

It is not hard to see that a direct product of groups is indeed a group.

In a group $\mathcal{G}$, there may be a subset of $\mathcal{G}$, which is itself a group w.r.t. the operation on $\mathcal{G}$ restricted to that subset:

Definition C.3 (Subgroup). *Let* $(\mathcal{G}, \circ_{\mathcal{G}})$ *be a group,* $U \subset \mathcal{G}$, *and* $\circ_U$ *the restriction of* $\circ_{\mathcal{G}}$ *to* $U \times U$. $(U, \circ_U)$ *is said to be a subgroup of* $(\mathcal{G}, \circ_{\mathcal{G}})$ *(written* $U \leqslant \mathcal{G}$*), if* $(U, \circ_U)$ *fulfills the group axioms. Let now* $\mathcal{G}$ *be written multiplicatively. One can define alternatively,* $U \leqslant \mathcal{G}$ *if:*

- $\forall x, y \in U$: $xy \in U$
- $\forall x \in U$: $x^{-1} \in U$
- $\forall x, y \in U$: $xy^{-1} \in U$

It is obvious that the neutral element of U coincides with that of G. Given a subset $X \subseteq \mathcal{G}$. With X, one can associate the following subgroup:

$$\langle X \rangle := \bigcap_{X \subseteq U \leqslant \mathcal{G}} U, \tag{C.1}$$

which is called the subgroup generated by X. One could also say: $\langle X \rangle$ is generated by X. If a group is generated by a finite subset, then this group is called finitely generated. A group is called cyclic if it is generated by a set of 1 element. A subgroup of a group $\mathcal{G}$ which is generated by non-empty $X \subset \mathcal{G}$ can be alternatively described as finite products of elements of $X \cup X^{-1}$:

$$\langle X \rangle = \{ \prod_{k \in [n]} x_k : n \in \mathbb{N},\ x_k \in X \cup X^{-1},\ \forall k \}. \tag{C.2}$$

In case $\mathcal{G}$ is abelian, the subgroup of $\mathcal{G}$, which is generated by finite subset $\{a_n\}$ of $\mathcal{G}$ can alternatively be written by:

$$\langle \{a_n\} \rangle = \{ \prod_n a_n^{\nu_n} : \nu_n \in \mathbb{Z},\ \forall n \}. \tag{C.3}$$

Furthermore, it is not hard to see that indeed:

$$\langle \{a_n\} \rangle = \prod_n \langle a_n \rangle. \tag{C.4}$$

A subgroup H of a group $\mathcal{G}$ is said to be normal if it commutes with all elements of $\mathcal{G}$, i.e. $gH = Hg$, $\forall g \in \mathcal{G}$.

Later, we shall often use the following notations:

Notations 5. *Let* $(\mathcal{G}, \circ)$ *be a group,* $A, B \subseteq \mathcal{G}$, $n \in \mathbb{N}_0$. *We define:*

- $A \circ B := \{a \circ b : a \in A, b \in B\}$
- $A^0 := \{e\}$
- $A^{-1} := \{a^{-1} : a \in A\}$
- $A^{n+1} := A^n \circ A$
- $A^{-n} := (A^n)^{-1}$

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be groups, both written multiplicatively. $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ is said to be a (group) homomorphism if $\forall g, h \in \mathcal{G}_1$:

$$\phi(gh) = \phi(g)\phi(h).$$

A (group) isomorphism is defined as a bijective group homomorphism, a (group) monomorphism as an injective group homomorphism, and a (group) epimorphism as a surjective group homomorphism. Some important properties of homomorphisms are summarized in the following lemma:

Lemma C.1. *Let $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ be a homomorphism between two groups. Then the following holds:*

- *Homomorphisms maps the identity to identity, i.e. $\phi(e_1) = e_2$, where e_1 is the identity in $\mathcal{G}_1$, and e_2 in $\mathcal{G}_2$.*
- *The image of a power under homomorphisms is again a power, in the sense that $\forall g \in \mathcal{G}_1,\ n \in \mathbb{Z} :\ \phi(g^n) = \phi(g)^n$.*
- *The image of a subgroup undera homomorphism is again a subgroup, i.e. $H \leqslant \mathcal{G}_1 \Rightarrow \phi(H) \leqslant \mathcal{G}_2$.*
- *The preimage of a subgroup under homomorphisms is again a subgroup, i.e. $H \leqslant \mathcal{G}_2 \Rightarrow \phi^{-1}(H) \leqslant \mathcal{G}_1$.*

The kernel of a group homomorphism $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ is defined as the set:

$$\ker(\phi) := \{h \in \mathcal{G}_1 : \phi(h) := e_2\}.$$

From above lemma, one gets immediately that $\ker(\phi) \leqslant \mathcal{G}_1$. A group homomorphism is said to be a (group) isomorphism if it is bijective.

Each $U \leqslant \mathcal{G}$ of abelian $\mathcal{G}$ induces the following set:

Definition C.4 ((left) Coset). *Let $U \leqslant \mathcal{G}$. Then, for $x, y \in \mathcal{G}$, $x \sim y \ :\Leftrightarrow xU = yU$ defines an equivalence relation on $\mathcal{G}$. For $x \in \mathcal{G}$, we define the set of equivalence class $[x] := xU$ of $x \in \mathcal{G}$ w.r.t. $\sim$ as the coset of U w.r.t. x.*

The right coset can be defined analogously, by taking the multiplication by elements of $\mathcal{G}$ from the right side. If there is a possibility for confusion, we write $[x]_U$ instead of $[x]$. Since we will later consider mostly Abelian groups, it is unnecessary to make a distinction between left and right cosets. All the preceding statements respective for general groups will be declared for the left operation, if not otherwise stated, the "right" version of the preceeding definition is to be analogously defined and the "right" version of the preceeding statements holds also .

The following property of cosets can be easily shown:

Lemma C.2. *For $U \leqslant \mathcal{G}$ and elements $a, b \in \mathcal{G}$, it holds:*

- $aU = U \Leftrightarrow a \in U$
- $aU = bU \Leftrightarrow a^{-1}b \in U$
- $aU \cap bU \neq \varnothing \Leftrightarrow aU = bU$

Clearly, a cosets aU, $a \in \mathcal{G}$, of a subgroup U, gives a partition of the group $\mathcal{G}$ in the way, s.t.: $\mathcal{G} = \bigcup_{a \in \mathcal{G}} aU$, where 2 cosets is either disjunct or equal. This suggests the following definition:

Definition C.5. *Let $U \leqslant G$. The quotient group $\mathcal{G}/U$ is defined as:*

$$\mathcal{G}/U := \{[x] : x \in \mathcal{G}\},$$

equipped with the multiplication $(xU, yU) \mapsto (xU)(yU) = xyU$.

In other words, a quotient group is the set of all equivalence classes of elements of $\mathcal{G}$ w.r.t. a subgroup U. It is obvious, that $\mathcal{G}/U$ with the operation: $(xU, yU) \mapsto (xU)(yU) = xyU$ is indeed a group. For some cases, it is convenient to define a representation system, in form of a subset $D \subseteq \mathcal{G}$ called (left) transversal, for the cosets in a quotient group $\mathcal{G}/U$, in the sense that $\forall [x] \in \mathcal{G}/U : \#(D \cap [x]) = 1$. The mapping which send each element of a group $\mathcal{G}$ to its equivalence class in $\mathcal{G}/U$, where $U \leqslant \mathcal{G}$, is called canonical homomorphism.

If not otherwise stated, $\mathcal{G}$ and $\mathcal{H}$, each written multiplicatively, stand in the following for groups.

C.2. Topological groups

As usual, the elements of a topology are defined as open sets. For ease of notations, most of the considered groups are written multiplicatively. If not otherwise stated, the results introduced in this section hold also for the other side of the operation. A notion which connects topological - and group idea is introduced in the following definition:

Definition C.6 (Topological Group). *Let $\mathcal{G}$ be a group written multiplicatively, which is also a topological space. $\mathcal{G}$ is said to be a topological group, if the multiplication $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(x, y) \mapsto xy$, and the inversion $\mathcal{G} \to \mathcal{G}$, $x \mapsto x^{-1}$ is continuous.*

A set U in a topological group is said to be symmetric, if $U = U^{-1}$. The following lemma gives some elementary properties of topological groups:

Lemma C.3. *Let $\mathcal{G}$ be a topological group, with identity e. Then the following statements hold:*

(a) For any $g \in \mathcal{G}$, the left - and the right translation by g are homeomorphism. Also the inversion is a homeomorphism.

(b) The neighborhoods of any point in $\mathcal{G}$ is fully described by the neighborhoods of the identity, i.e. $U \subseteq \mathcal{G}$ is a neighborhood of $g \in \mathcal{G}$ if and only if $g^{-1}U$ is a neighborhood of e.

(c) All neighborhoods of the identity is closed under inversion, i.e. if U is a neighborhood of e then U^{-1} is a neighborhood of e. Furthermore, every neighborhood U of the identity contains a symmetric neighborhood of the identity, viz. $U \cap U^{-1}$

(d) Given a neighborhood U of the identity e. Then there exists a neighborhood V of e s.t. $V^2 \subseteq U$

(e) If $A, B \subseteq \mathcal{G}$ are compact, then AB is compact.

(f) If $A \subseteq \mathcal{G}$ is open and B is arbitrary, then AB is also open.

(g) If $A \subseteq \mathcal{G}$ is closed and $K \subseteq \mathcal{G}$ is compact, then AK is closed.

the statements (a), (b), and (c) in above lemma are immediate consequences of continuity of the multiplication and inversion of elements of a topological group. The proof of statement (d) can be found e.g. in (4.5) Theorem in [24]. The proof of statements (e), (f), and (g) can be found e.g. in (4.4) Theorem in [24]. From statement (e) and by induction, one can easily see that compact subsets of a topological group is closed under finite multiplication. From statement (f) and by induction, one gets also that open subsets are closed under finite multiplication. It is easy to see that the statement (d) can be generalized as follows: For each neighborhood U of the identity, and $k \in \mathbb{N}$, there exists an neighborhood V of the identity s.t. $V^k \subseteq U$. Furthermore, from (c) and the definition of neighborhood, one can assume V, if required, to be open, symmetric, (logical-)or compact.

Given a symmetric neighborhood of the identity of a topological group. One can construct a subgroup of this topological group as follows:

Proposition C.4. *Let U be a symmetric neighborhood of the identity e of a topological group $\mathcal{G}$. Then $H = \bigcup_{n\in\mathbb{N}} V^n$ is an open (and hence also closed) subgroup of $\mathcal{G}$*

Proof. Let $n, m \in \mathbb{N}$, and U a symmetric neighborhood of e. For $x \in U^n$ and $y \in U^m$, it clearly holds $xy \in U^{n+m}$, and also $x^{-1} \in U^n$, since U is symmetric. Hence H is a subgroup of $\mathcal{G}$. To show that H is open, take an arbitrary $h \in H$. h is contained in U^n, for some $n \in \mathbb{N}$. For the neighborhood hU of U, we obtain: $hU \subseteq U^{n+1} \subseteq$. So, for each $h \in H$, there exist a neighborhood, e.g. hU, contained in H, and correspondingly H is open. From prop. H is also closed. ∎

We give in the following the notions of mapping between topological groups:

Definition C.7 (Continuous Homomorphism and Topological Group Isomorphism). *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be topological groups. A map $f : \mathcal{G}_1 \to \mathcal{G}_2$ is said to be a continuous homomorphism (or topological group homomorphism), if it is a group homomorphism and continuous. An injective (resp. surjective) topological group homomorphism is called a topological group monomorphism (resp. epimorphism). If f is both, a group isomorphism, and a homeomorphism, then it is said to be a topological group isomorphism. f is said to be a topological group embedding, if f is a topological group isomorphism between the domain - and the range of f.*

In most cases, when considering mapping between topological groups, we shall say simply write homomorphism (resp. isomorphism, monomorphism, embedding) instead of topological group homomorphism (resp. - isomorphism, monomorphism, embedding). Notice the difference of this terms with the group theoretic - and topological terms of the same name. When speaking about topological groups, we shall further call the latter terms as algebraic homomorphism (resp. - isomorphism, - monomorphism, -epimorphism), and topological embedding.

By the fact that the restriction of continuous mapping (in particular the group operation) is itself continuous w.r.t. the relative topology, one obtains immediately the following statement:

Proposition C.5. *Let $\mathcal{G}$ be a topological group, and H be a subgroup of $\mathcal{G}$. Equipped with the topology relative to $\mathcal{G}$, H becomes itself a topological group.*

If not otherwise stated, we shall always equip subgroup of a topological group with the subspace topology. A subgroup H of a topological group $\mathcal{G}$ equipped with the subspace topology inherited from $\mathcal{G}$ is automatically closed in $\mathcal{G}$:

Proposition C.6. *Let $\mathcal{G}$ be a topological group, and H a subgroup of $\mathcal{G}$. If H is open, then H is also closed.*

Proof. Let $\{x_i\}_{i\in\mathcal{I}}$ be a (right) transversal of H in $\mathcal{G}$. By openness assumption of H, it follows immediately that each $\{Hx_i\}_{i\in\mathcal{I}}$ is open. Clearly for a $i_0 \in \mathcal{I}$, $Hx_{i_0} = H$. Hence, one can write $\mathcal{G}$ as the disjoint union $\mathcal{G} = H \bigcup_{i\in\mathcal{J}} Hx_i$, where $\mathcal{J} := \mathcal{I}\backslash\{i_0\}$. So, in $\mathcal{G}$, $\mathcal{H}$ is complement of an open set, and therefore closed. ∎

The following proposition asserts that a topological group possessess a "good-natured" structure, since its topological informations can be induced from the topological structure of the neighborhoods of the identity, and since continuity of a homomorphism can be induced from the its continuity at the identity.

Proposition C.7. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be topological groups, and let $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ be a group homomorphism. Then the following holds:*

(a) Let $x \in \mathcal{G}_1$ (In particular $x = e_1$, where e_1 is the identity of $\mathcal{G}_1$), $\{gU : g \in \mathcal{G}_1,\ U \in \mathcal{N}_{\mathcal{G}_1}\}(x)$, where $\mathcal{N}_{\mathcal{G}_1}(x)$ denotes the set of all neighborhoods x in $\mathcal{G}_1$, is exactly the topology of $\mathcal{G}_1$

(b) ϕ is continuous if and only if ϕ is continuous at any point in $\mathcal{G}_1$, in particular at the identity of $\mathcal{G}_1$.

Specifically, one obtains the following characterizations of the structure of a topological group induced from its local property:

Proposition C.8. *Let $\mathcal{G}$ be a topological group. Then the following holds:*

- *$\mathcal{G}$ is locally compact if and only if there is one point of $\mathcal{G}$ possessing a neighborhood base of compact sets.*
- *$\mathcal{G}$ is locally (path) connected if and only if there is one point of $\mathcal{G}$ possessing a neighborhood base of open, (path) connected sets.*
- *$\mathcal{G}$ is locally euclidean if and only if there is one point of $\mathcal{G}$ with a neighbourhood homeomorphic to an open subset of $\mathbb{R}^N$.*
- *$\mathcal{G}$ is discrete if and only if there is one point of $\mathcal{G}$ which is isolated.*

The following proposition gives among others the behaviour of a topological group under quotient:

Proposition C.9. *Let N be a normal subgroup of a topological group $\mathcal{G}$. Equipping the quotient group the quotient topology induced by the canonical quotient homomorphism $\mathrm{q} : \mathcal{G} \to \mathcal{G}/N$, then the following holds:*

(a) $\mathcal{G}/N$ becomes a topological group, and q is under this circumstances continuous and open.

(b) $\mathcal{G}/N$ is discrete, if and only if N is open.

(c) If N is in addition compact, then q is also a closed map.

Now, let $\mathcal{G}_1$ be an another topological group, and $f : \mathcal{G}/N \to \mathcal{G}_1$ be an algebraic homomorphism. Then f is continuous, if and only if $f \circ \mathrm{q} : \mathcal{G} \to \mathcal{G}_1$ is continuous.

In particular, we shall always equip quotients of topological groups of the form $\mathcal{G}/N$, where N is normal, and shall always see $\mathcal{G}/N$ as a topological group. The following proposition, asserts in some sense the invariant of locally compactness under quotient:

Proposition C.10. *Let $\mathcal{G}$ be a locally compact group, $H \leqslant \mathcal{G}$ a closed normal subgroup of $\mathcal{G}$, and $\mathrm{q} : \mathcal{G} \to \mathcal{G}/H$ the canonical projection. Then the following statements holds:*

(a) $\mathcal{G}/H$ is locally compact

(b) For a compact subset $B \subseteq \mathcal{G}/H$ there exist always a compact subset $K \subseteq \mathcal{G}/H$, s.t. $p(K) = C$.

The following proposition is due to Frobenius, which constitute an analogy to the useful First Isomorphism Theorem of group theory:

Proposition C.11. *Let $\mathcal{G}$ and $\tilde{\mathcal{G}}$ be topological groups, and let $f : \mathcal{G} \to \tilde{\mathcal{G}}$ be an epimorphism. Consider the canonical homomorphism $\mathrm{q} : \mathcal{G} \to \mathcal{G}/\ker f$, then the unique homomorphism $\tilde{f} : \mathcal{G}/\ker f \to \tilde{\mathcal{G}}$, for which $f = \tilde{f} \circ \mathrm{q}$ holds, is a continuous algebraic isomorphism. Furthermore, f_1 is an (topological group) isomorphism, if and only if f is open.*

In contrast to the First Isomorphism Theorem, found in group theory, openness requirement on f is necessary to "factorize" f through quotient. Above proposition can easily shown by involving basic properties of open maps and quotient of topological groups. The following easy application of above proposition might be helpful for later approaches. In particular, it say roughly that quotient of topological groups are invariant under topological group isomorphism:

Corollary C.12. *Let $\mathcal{G}$ and $\mathcal{G}_1$ be topological groups, and $f : \mathcal{G} \to \mathcal{G}_1$ an isomorphism between them. For each normal subgroup $N \leqslant \mathcal{G}$, $f(\mathcal{G})/f(N) = \mathcal{G}_1/f(N)$ is top. group isomorphic to $\mathcal{G}/N$.*

Proof. It can easily be shown that $f(N)$ is a normal subgroup of $\mathcal{G}_1$. Accordingly, the canonical quotient algebraic homomorphism $\mathrm{q} : \mathcal{G}_1 \to \mathcal{G}_1/f(N)$ is continuous, surjective and open by prop. C.9. As a composition of two continuous, surjective and open algebraic homomorphism, $\tilde{f} := \mathrm{q} \circ f$ fulfills also the properties. Furthermore, it holds $\ker \tilde{f} = \ker \mathrm{q} = N$. Hence, by prop. C.11, the desired statement holds. ■

Lemma C.13. *Let be $\mathcal{G}$ a locally compact group. If $\mathcal{G}$ is compactly generated, then there exists a relatively compact neighborhood of the identity which generates $\mathcal{G}$.*

It can be shown, that the product of arbitrary number of topological groups equipped with suitable topology is also a topological group:

Proposition C.14. *Let $\{\mathcal{G}_i\}_{i\in\mathcal{I}}$ be a family of topological groups. The product $\mathcal{G} := \prod_{i\in\mathcal{I}} \mathcal{G}_i$, equipped with the product topology, is also a topological groups.*

Considering product of topological groups, we shall always equip it with product topology, and see it as a topological group.

Bibliography

[1] E. Agora, J.Antezana, and C. Cabrelli. Multi-tiling sets, Riesz Bases, and Sampling near the Critical Density in LCA Groups. *pre-print arXiv*, Apr. 2014.

[2] R. Balan, B. G. Bodemann, P. G. Casazza, and D. Edidin. Painless reconstruction from magnitudes of frame coefficients. *J. Fourier Anal. Appl.*, 15(4):488 – 501, 2009.

[3] R. Balan, P. G. Casazza, and D. Edidin. On signal reconstruction without phase. *Appl. Comput. Harmon. Anal.*, 20(3):345 – 356, May 2006.

[4] H. Boche and B. Farrell. Strong divergence of reconstruction procedures for the paley–wiener space and the hardy space. *Journal of Approximation Theory*, 183(0):98 – 117, 2014.

[5] H. Boche and U.J. Monich. Sampling of deterministic signals and systems. *Signal Processing, IEEE Transactions on*, 59(5):2101–2111, May 2011.

[6] Nicolas Bourbaki. *Elements of mathematics. General topology. Part 1.* Hermann, 1966.

[7] E. J. Candès, Y. C. Eldar, T. Strohmer, and V. Voronski. Phase retrieval via matrix completion. *Siam J. Imaging Sci.*, 6(1):199 – 225, 2013.

[8] O. Christensen. *An Introduction to Frames and Riesz Bases.* Boston, MA: Birkhäuser, 2003.

[9] A. Deitmar and S. Echterhoff. *Principles of Harmonic Analysis.* Springer Publishing Company, Incorporated, 2009.

[10] J. Elstrodt. *Mass- und Integrationstheorie.* Springer-Lehrbuch, 6.te, korrigierte auflage edition, 2009.

[11] C. Falldorf, M. Agour, C. v. Kopylow, and R. B. Bergmann. Phase retrieval by means of spatial light modulator in the fourier domain of an imaging system. *Applied Optics*, 49(10):1826 – 1830, April 2010.

[12] J. Feldmann and E. F. Greenleaf. Existence of borel transversals on groups. *Pacific Journal of Math.*, 25(3):455 – 461, 1968.

[13] P. Feng and Y.Bresler. Spectrum-blind Minimum-rate Sampling and Reconstruction of Multiband Signals. *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, 3:1688 – 1691, May 1996.

[14] D.H. Fremlin. *Measure Theory*, volume 1. Biddles Short Run Books, 3.ed. edition, 2004.

[15] B. Fuglede. Commuting self-adjoint partial differential operators and a group theoretic problem. *Journal of Functional Analysis*, 16(1):101 – 121, 1974.

[16] N. T. Gaarder. A Note on the Multidimensional Sampling Theorem. *Proc. IEEE*, 60:247 – 248, Feb. 1972.

[17] S. Grepstad and N. Lev. “multi-tiling and Riesz bases. *Adv. Math.*, 252:1–6, Feb. 2014.

[18] K. Groechenig. *Foundations of Time-Frequency Analysis.* Birkhäuser, 2001.

[19] K. Groechenig, G. Kutyniok, and K. Seip. Landau's Necessary Density Conditions for LCA groups. *J. Funct. Anal.*, 255:1831–1850, 2008.

[20] P. Gruber. *Convex and Discrete Geometry (Grundlehren der mathematischen Wissenschaften).* Springer, softcover reprint of hardcover 1st ed. 2007 edition, 2010.

[21] M. H. Hayes, J. S. Lim, and A.V. Oppenheim. Signal reconstruction from phase or magnitude. *IEEE Trans. Acoust., Speech, Signal Process.*, ASSP-28(6):672 – 680, Dec. 1980.

[22] C. Heil. *A Basis Theory Primer.* Birkhaeuser, 2010.

[23] C. Herley and P. W. Wong. Minimum rate sampling and reconstruction of signals with arbitrary frequency support. *IEEE Trans. Inf. Theory*, 45(5):1555 – 1564, Jul. 1999.

[24] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis. Vol. 1: Structure of Topological Groups; Integration Theory; Group Representations. 2nd ed.* Springer-Verlag, 1979.

[25] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis. Vol. 2: Structure and Analysis for Compact Groups; Integration Theory; Group Representations.* Springer-Verlag, 1979.

[26] A. Iosevich, N. Katz, and S. Pedersen. Fourier bases and a distance problem of erdös. *Math. Res. Lett.*, 6(2):251 – 255, 1999.

[27] A. Iosevich, N. Katz, and T. Tao. Convex bodies with a point of curvature do not have fourier bases. *Amer. J. Math.*, 123(1):115 – 120, 2001.

[28] A. Iosevich, N. Katz, and T. Tao. The fuglede spectral conjencture holds for convex planar domains. *Amer. J. Math.*, 10(5 - 6):559 – 569, 2003.

[29] P. Jaming. The phase retrieval problem for the radar ambiguity function and vice versa. *IEEE Intern. Radar Conf.*, May 2010.

[30] E. Kaniuth and G. Kutyniok. Zeros of the zak transform on locally compact abelian groups. *Proc. of the Am. Math. Soc.*, 126(1):3561 – 3569, 1998.

[31] I. Kluvanek. Sampling theorem in abstract harmonic analysis. *Nat. -Fyz. Casopis Sloven. Akad.*, 15:43 – 48, 1965.

[32] M. Kolountzakis. Non-symmetric convex domains have no basis of exponentials. *Illinois J. Math.*, 44:542 – 550, 2000.

[33] M. Kolountzakis. Multiple Lattice Tiles and Riesz Bases of Exponentials. *Proc. Amer. Math. Soc.*, 143(2):741 – 747, Feb. 2015.

[34] W. Kotelnikow. On the transmission capacity of „ether" and wire in electro communications. *Izd. Red. Upr. Svyazzi RKKA.*, 1933.

[35] H. J. Landau. Necessary density conditions for sampling and interpolation of certain entire functions. *Acta Math.*, 117:37 – 52, 1967.

[36] Y. P. Lin and P. P. Vaidyanathan. Periodically nonuniform sampling of bandpass signals. *IEEE Trans. Circuits Syst. II*, 45(3):340 – 351, Mar. 1998.

[37] B. Matei and Y. Meyer. Quasicrystals are sets of stable sampling. *C. R. Acad. Sci. Paris Ser. I Math*, 346:1235 –1238, 2008.

[38] B. Matei and Y. Meyer. Simple quasicrystals are sets of stable sampling. *Complex Var. Elliptic Equ.*, 55(8-10):947–964, 2010.

[39] M. Mishali and Y. Eldar. Blind multiband signal reconstruction: Compressed sensing for analog signals. *IEEE Transactions on Signal Processing*, 57(3):993–1009, 2009.

[40] M. Mishali and Y. Eldar. From theory to practice: Sub-nyquist sampling of sparse wideband analog signals. *J. Sel. Topics Signal Processing*, 4:375–391, 2010.

[41] U. Monich and H. Boche. A Two Channel Approach for System Approximation with General Measurement Functionals. *International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2015. To appear.

[42] Sidney A. Morris. *Pontryagin duality and the structure of locally compact abelian groups.* Cambridge University Press, Cambridge, 1977.

[43] H. Nyquist. Certain topics in telegraph transmission theory. *AIEE Trans.*, 47(1):617–644, 1928.

[44] E. Parzen. A Simple Proof and Some Extensions of Sampling Theorems. Technical report, Standford University, Standford, CA, 1956. Tech. Rep. 7.

[45] R. Pedarsani, K. Lee, and K. Ramchandran. PhaseCode: Fast and Efficient Compressive Phase Retrieval based on Sparse-Graph-Codes. *Pre-print arXiv*, July 2014.

[46] V. Pohl and H. Boche. U-Invariant Sampling and Reconstruction in Atomic Spaces with Multiple Generators. *IEEE Trans. Signal Process.*, 60(7):3506–3519, Jul. 2012.

[47] V. Pohl and H. Boche. Sampling Theory and Frames. University Lecture, 2013.

[48] V. Pohl, F. Yang, and H. Boche. Phase retrieval from low rate samples. *To appear.*

[49] H. Raabe. Untersuchungen an der wechselzeitigen mehrfachuebertragung (multiplexuebertragung). *Elektr. Nachr.*, 16:213 – 228, 1939.

[50] G. Ross, M. A. Fiddy, M. Nieto-Vesperinas, and M. W. L. Wheeler. The phase problem in scattering phenomena: The zeros of entire functions and their significance. *Proc. R. Soc. Lond.*, A 360(1700):25 – 45, Mar. 1978.

[51] Walter Rudin. *Fourier analysis on groups.* Interscience tracts in pure and applied mathematics. Interscience Publishers, New York, London, 1962. Autre tirage : 1967.

[52] C. Shannon. Communication in the presence of noise. *Proc. IRE*, 37(1), 1949.

[53] G. Thakur. Reconstruction of bandlimited functions from unsigned samples. *J. Fourier Anal. Appl.*, 17(4):720 – 732, Aug. 2011.

[54] R. Venkataramani and Y. Bresler. Perfect reconstruction formulas and bounds on aliasing error in sub-nyquist nonuniform sampling of multiband signals. *IEEE Trans. Inf. Theory*, 46(6):2173 – 2183, Sep. 2000.

[55] P. Walters. *An Introduction to Ergodic Theory.* Graduate Texts in Mathematics. Springer New York, 2000.

[56] Dirk Werner. *Funktionalanalysis.* Springer, 6. auflage edition, Sep. 2007.

[57] J. Whittaker. The fourier theory of the cardinal functions. *Proc. Edinburgh Math. Soc.*, 1, 1929.

[58] D. Wieruch and V. Pohl. A Cognitive Radio Architecture based on sub-Nyquist Sampling. *Proc. 5th IEEE Intern. Symposium on New Frontiers in Dynamic Spectrum Access Networks (DySPAN)*, May. 2011.

[59] X. Xiao and Q. Shen. Wave propagation and phase retrieval in fresnel diffraction by a distorted-object approach. *Phys. Rev.*, B 72, 2005.

[60] F. Yang, V. Pohl, and H. Boche. Phase retrieval via structured modulations in paley wiener spaces. *10th Intern. Conf. on Sampling Theory and Applications (SampTA)*, July 2013.

[61] Y.Bresler and P. Feng. Spectrum-blind Minimum-rate Sampling and Reconstruction of 2-d Multiband Signals. *Proc. IEEE Int. Conf. Image Process.*, 1:701 – 704, Sep. 1996.

[62] G. Zauner. *Quantendesigns –Grundzuege einer nichtkommutativen Designtheorie.* PhD thesis, University of Vienna, 1999.

[63] F. Zhang, G. Pedrini, and W. Osten. Phase retrieval of arbitrary complex-valued fields through aperture-plane modulation. *Phys. Rev.*, A 75, 2007.